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TOPICS IN THE THEORY OF GENERALISED HYPERGEOMETRIC SERIES.

A THESIS SUBMITTED FOR
THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN
THE UNIVERSITY OF DURHAM.

BY

ARTHUR LAKIN.

DECEMBER 1950.

CONTENTS

It will be obvious to anyone reading this thesis how far the writer has been influenced by the numerous papers on this subject by J. L. Burchnall and T. W. Chaundy. In particular he wishes to acknowledge a deep debt of gratitude to Professor Burchnall who has supervised this work.

CHAPTER I.

GENERALISED HYPERGEOMETRIC SERIES. METHODS OF OBTAINING TRANSFORMATIONS OF SERIES.

1.1. INTRODUCTION.

The series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{n!\,(b_1)_n (b_2)_n \cdots (b_q)_n} x^n ,$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$,

is called a generalised hypergeometric series and is denoted by $_pF_q\left[\begin{matrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_q \end{matrix}; x\right]$ or where no confusion can arise by $F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right]$.

In what follows we will be concerned only with the case $p = q+1$. Then the series converges when $|x| < 1$, when $x = 1$ provided $R(\Sigma b - \Sigma a) > 0$, and when $x = -1$ provided that $R(\Sigma b - \Sigma a + 1) > 0$.

If $\delta \equiv x\frac{d}{dx}$ then, since $\delta x^n = n x^n$,

(1) $$\delta F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right] = x\frac{\Pi(a_r)}{\Pi(b_s)} F\left[\begin{matrix} a_r+1 \\ b_s+1 \end{matrix}; x\right],$$

where in the series on the right every parameter has been increased by unity, and

(2) $$(\delta + a_r) F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right] = a_r F\left[\begin{matrix} a_r+1 \\ b_s \end{matrix}; x\right]$$

(3) $$(\delta + b_s - 1) F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right] = (b_s - 1) F\left[\begin{matrix} a_r \\ b_s - 1 \end{matrix}; x\right]$$

where only the parameter specialised is augmented or diminished by unity on the right hand side. From equations (1) – (3) we may readily deduce the differential equation satisfied by $F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right]$, namely

(4) $$\left\{ \delta \prod_b (\delta + b_s - 1) - x \prod_a (\delta + a_r) \right\} y = 0.$$

If we set $y = x^{1-b_k} z$, and use the further property of the δ-operator $f(\delta)\, x^c z = x^c f(\delta + c)\, z$, then z satisfies the equation

(5) $$\left\{ \delta(\delta + 1 - b_k) \prod{}' (\delta + b_s - b_k) - x \prod (\delta + 1 + a_r - b_k) \right\} z = 0,$$

where the dash indicates that the factor $\delta + b_k - b_k$, i.e. δ, has been taken outside the product sign. Comparison with (4) shows that (5) has the solution $z = F\left[\begin{matrix} 1 + a_r - b_k \\ 2 - b_k,\ 1 + b_s - b_k \end{matrix}; x\right]$, and therefore that (4) has the solutions

$$y = x^{1-b_k} F\left[\begin{matrix} 1 + a_r - b_k \\ 2 - b_k,\ 1 + b_s - b_k \end{matrix}; x\right].$$

1.2. HISTORICAL NOTE.[1]

The term 'hypergeometric' was first used by Wallis in 1655 to describe the series whose $(n+1)^{th}$ term is $a(a+b)\cdots(a+nb)$. The series $\sum \frac{(a)_n (b)_n}{n!\,(c)_n} x^n$ appears to have been introduced into analysis by Gauss in 1812 and given its present name by Kummer in 1836. It was Kummer who obtained the twenty four solutions of the differential equation

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(1) The information in the first part of this note is taken from the 'Encyclopädie der Mathematischen Wissenschaften' Bd. 2. 2. p 537 et seq., where references will be found to early papers; see also the opening paragraphs of Barnes 1.

satisfied by the general ${}_2F_1$, and showed that these twenty four solutions are reducible to six sets of four, each four being identical functions differently expressed. The six sets can be divided into pairs, each pair corresponding to one of the three singularities $0, 1, \infty$, of the differential equation. Riemann in 1857 extended the theory by introducing his P-function and discussed the general theory of transformation of the variable. Thomae in 1879 worked out in detail, from the theory of linear differential equations, the relations which connect any one of the twenty four solutions of the equation with the two essentially different solutions valid in the neighbourhood of either of the singularities not associated with the particular solution chosen. This was a task begun by Kummer and completed by Goursat. Riemann and Jacobi working with definite integrals showed that, subject to certain conditions,

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt,$$

and Klein considered similar integrals taken round Pochhammer circuits.

In 1907 Barnes developed the whole theory 'de novo' by considering contour integrals involving gamma functions of the variable of integration, a typical integral being

$$\frac{1}{2\pi i}\int \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-x)^s\,ds, \qquad |\arg(-x)| < \pi,$$

where the contour is parallel to the imaginary axis with loops if necessary to ensure that the points $0, 1, 2, \cdots,$ are to the right, and the points $-a, -a-1, \ldots, -b, -b-1, \cdots$, are to the left of the contour, which is completed by an infinite semicircle. Evaluating the integral by summing the residues at the poles to the right of the imaginary axis gives rise to the series ${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right]$. Whittaker and Watson adopt this treatment in the later editions of 'Modern Analysis'.

Meanwhile Clausen in 1828 was the first to consider generalised series with more than three parameters. Thomae in 1879 obtained a large number of relations between two and three series of the type ${}_3F_2$ with unit argument, and other identities concerning series with arguments ± 1 were discovered, the most important of these being first published in papers by Saalschütz 1890, Dixon 1903, and Dougall 1907. These results were largely rediscovered by Ramanujan and were collected together by Hardy in a paper entitled 'A chapter from Ramanujan's note book' in 1923.[1] This paper seemed to stimulate further work on generalised series; a number of new identities were obtained mainly by Bailey and Whipple and all of them are summarised in the Cambridge tract 'Generalised Hypergeometric Series' by Bailey, published in 1935.

1.3. OTHER GENERALISATIONS OF HYPERGEOMETRIC SERIES.

The hypergeometric series has been generalised in other ways. Heine in his 'Theorie der Kugelfunctionen'. I.(1878) pps. 97–125, has considered the basic series

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(1) Hardy 1.

$$\underset{2}{\Phi}_{1}\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = 1 + \frac{(1-a)(1-b)}{(1-q)(1-c)}x + \frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2 + \cdots ,$$

and transformation analogous to those connecting ordinary generalised series have been obtained by Jackson, Watson and Bailey and are collected together in chapter VIII of Bailey's tract. Such series will be considered further in chapter III of this thesis.

Further generalisations can be obtained by increasing the number of variables. In the first part of the treatise 'Fonctions hypergéométriques et Hypersphériques' Appell and Kampé de Fériet consider four distinct series

$$F_{(1)}\left[\begin{matrix} a; \\ c; \end{matrix} b, b'; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n ,$$

$$F_{(2)}\left[\begin{matrix} a; & b, b' \\ & c, c' \end{matrix}; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_m(c')_n} x^m y^n ,$$

$$F_{(3)}\left[\begin{matrix} & a, a'; b, b' \\ c; & \end{matrix}; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n ,$$

$$F_{(4)}\left[\begin{matrix} a, b; & \\ & c, c' \end{matrix}; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(c')_n} x^m y^n .$$

In addition to the transformations of Appell series collected in chapter IX of Bailey's tract, Burchnall and Chaundy[1] have obtained a number of transformations of such series into extended hypergeometric series, i.e. series which themselves contain hypergeometric series in the general term.[2] Horn, considering solutions of the partial differential equations satisfied by Appells series has

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[1] Burchall and Chaundy. 1.2. [2] Chaundy. 2.

[3] Horn. 1. 2. 3.

introduced a number of related series of which the following two are typical and will be needed later,

$$G_{(2)}(a,a',b,b';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_{n-m}(b')_{m-n}}{m!\,n!} x^m y^n,$$

$$H_{(2)}(a;b,b',c';d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m-n}(b)_m(b')_n(c')_n}{m!\,n!\,(d)_m} x^m y^n,$$

where $(a)_{-m} = \dfrac{(-)^m}{(1-a)_m}$. Appells series are considered further in chapter IV of this thesis whilst examples of extended series occur in chapter V.

More recently Bailey[1] has considered series which are infinite in both directions, e.g.

$${}_2H_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; x\right] = \sum_{n=-\infty}^{+\infty} \frac{(a)_n(b)_n}{(c)_n(d)_n} x^n, \qquad |x| = 1.$$

Such series may be considered as the sum of two ordinary hypergeometric series

$${}_2H_2\left[\begin{matrix} a, b \\ c, d \end{matrix}; x\right] = {}_3F_2\left[\begin{matrix} 1, a, b \\ c, d \end{matrix}; x\right] + \frac{(1-c)(1-d)}{(1-a)(1-b)}\cdot\frac{1}{x}\,{}_3F_2\left[\begin{matrix} 1, 2-c, 2-d \\ 2-a, 2-b \end{matrix}; \frac{1}{x}\right],$$

and, in fact, both the series on the right are solutions of the same differential equation,

$$\delta\{(\delta+c-1)(\delta+d-1) - x(\delta+a)(\delta+b)\}y = 0,$$

a property which will be used in the next chapter to obtain some of Baileys results concerning bilateral series. M. Jackson[2] has recently somewhat extended Bailey's results and has also considered bilateral basic series.

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(1) Bailey. 2.

(2) Jackson. M. 1. 3. 4. also BAILEY. 5. 6. 7. 8. 9.

1.4. REDUCIBLE SERIES. TRANSFORMATIONS OF SERIES.

We say that a hypergeometric series with some fixed argument, usually ± 1, is reducible if it can be expressed in terms of gamma functions. The most important identities of this type are contained in the following theorems,

(1) Gauss' theorem, $${}_2F_1\left[\begin{matrix}a, b;\\ c\end{matrix}\right]^{(1)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

(2) Kummer's theorem, $${}_2F_1\left[\begin{matrix}2a_1, a_1+a_2\\ 1+a_1-a_2\end{matrix}; -1\right] = \frac{\Gamma(1+a_1-a_2)\Gamma(1+a_1)}{\Gamma(1+2a_1)\Gamma(1-a_2)},$$

(3) Saalschütz theorem,
$${}_3F_2\left[\begin{matrix}a_1, a_2, -n\\ b_1, 1+a_1+a_2-b_1-n\end{matrix};\right] = \frac{(b_1-a_1)_n(b_1-a_2)_n}{(b_1)_n(b_1-a_1-a_2)_n},$$

(4) Dixon's theorem,
$${}_3F_2\left[\begin{matrix}2a_1, a_1+a_2, a_1+a_3\\ 1+a_1-a_2, 1+a_1-a_3\end{matrix};\right] = \frac{\Gamma(1+a_1)\Gamma(1+a_1-a_2)\Gamma(1+a_1-a_3)\Gamma(1-a_1-a_2-a_3)}{\Gamma(1+2a_1)\Gamma(1-a_2)\Gamma(1-a_3)\Gamma(1-a_2-a_3)},$$

(5) Dougall's theorem
$${}_7F_6\left[\begin{matrix}2a_1, & 1+a_1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5, & -m\\ & a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5, & 1+2a_1+m\end{matrix};\right]$$
$$= \frac{(1+2a_1)_m(1-a_2-a_3)_m(1-a_2-a_4)_m(1-a_3-a_4)_m}{(1+a_1-a_2)_m(1+a_1-a_3)_m(1+a_1-a_4)_m(1-a_1-a_2-a_3-a_4)_m},$$

subject to the condition $a_2+a_3+a_4+a_5 = m+1$.

In the theorems of Saalschütz and Dougall the series terminate due to the presence of a negative integer as a numerator parameter. Also in Saalschütz theorem the sum of the denominator parameters exceeds by unity the sum of the numerator parameters. Series of any

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(1) Here, and in future, where the argument is equal to unity, it will be omitted. References to the original papers in which these theorems occur will be found in Bailey's tract.

order which exhibit this relation between the parameters are called 'Saalschützian'. In the theorems of Kummer, Dixon, and Dougall the numerator and denominator parameters can be arranged in pairs, each pair having the same sum which exceeds by unity the value of the remaining numerator parameter. Such series are called 'well-poised'.[1]

Identities have also been obtained connecting two or more series. The most important transformations of this type are the Thomae relations of which the following two are typical,

$$(6)\quad {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right] = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(s)}{\Gamma(a_1)\Gamma(s+a_2)\Gamma(s+a_3)}\, {}_3F_2\left[\begin{matrix} b_1-a_1, b_2-a_1, s \\ s+a_2, s+a_3 \end{matrix};\right],$$

where $s = b_1+b_2-a_1-a_2-a_3$,

$$(7)\quad {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right] = \frac{\Gamma(1-a_1)\Gamma(b_1)\Gamma(b_2)\Gamma(a_3-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)\Gamma(1+a_2-a_1)\Gamma(a_3)}\, {}_3F_2\left[\begin{matrix} a_2, 1+a_2-b_1, 1+a_2-b_2 \\ 1+a_2-a_3, 1+a_2-a_1 \end{matrix};\right]$$

+ a similar expression with a_2 and a_3 interchanged,

Whipple's relation connecting a 'well-poised' ${}_7F_6$ with a Saalschützian ${}_4F_3$

$$(8)\quad {}_7F_6\left[\begin{matrix} 2a_1, 1+a_1, a_1+a_2, a_1+a_3, a_1+a_4, a_1+a_5, a_1+a_6 \\ a_1, 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5, 1+a_1-a_6 \end{matrix};\right]$$

$$= \frac{\Gamma(1+a_1-a_4)\Gamma(1+a_1-a_5)\Gamma(1+a_1-a_6)\Gamma(1-a_1-a_4-a_5-a_6)}{\Gamma(1+2a_1)\Gamma(1-a_4-a_5)\Gamma(1-a_5-a_6)\Gamma(1-a_4-a_6)}$$

$${}_4F_3\left[\begin{matrix} 1-a_2-a_3, a_1+a_4, a_1+a_5, a_1+a_6 \\ 1+a_1-a_2, 1+a_1-a_3, a_1+a_4+a_5+a_6 \end{matrix};\right],$$

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[1] The terms 'Saalschützian' and 'well-poised' are due to Whipple; tract p. 11. The form of the parameters used here for 'well-poised' series is due to Burchnall. 1. ; it has the advantage of exhibiting the symmetry in the differential equations satisfied by such series.

which is valid provided that the series on the right terminates and the series on the left converges; and finally Bailey's relation connecting two 'well-poised' ${}_9F_8$,

$$(9)\quad {}_9F_8\left[\begin{matrix} 2a_1, 1+a_1, a_1+a_2, a_1+a_3, a_1+a_4, a_1+a_5, a_1+a_6, a_1+a_7, -m \\ a_1, 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5, 1+a_1-a_6, 1+a_1-a_7, 1+2a_1+m \end{matrix};\right]$$

$$= \frac{(1+2a_1)_m (1+2k-a_1-a_5)_m (1+2k-a_1-a_6)_m (1+2k-a_1-a_7)_m}{(1+2k)_m \ (1+a_1-a_5)_m \ (1+a_1-a_6)_m \ (1+a_1-a_7)_m}$$

$${}_9F_8\left[\begin{matrix} 2k, 1+k, k+a_2-a_1, k+a_3-a_1, k+a_4-a_1, a_1+a_5, a_1+a_6, a_1+a_7, -m \\ k, 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+2k-a_1-a_5, 1+2k-a_1-a_6, 1+2k-a_1-a_7, 1+2k+m \end{matrix};\right],$$

where $2k = 1+a_1-a_2-a_3-a_4$, and the parameters are subject to the restriction $a_2+a_3+a_4+a_5+a_6+a_7 = m+2$. These conditions imply that in both series the sum of the denominator parameters exceeds by two the sum of the numerator parameters.

1.5. METHODS OF OBTAINING TRANSFORMATIONS. The methods which have been principally used to obtain transformations of hypergeometric series are four in number. These methods appear to have been thoroughly exploited and form the subject matter of chapters II – VII of Bailey's tract. In the remainder of this chapter we will illustrate these methods by simple examples, and consider one or two additional ones. In the next chapter a new method will be described.

1.6. EQUATING COEFFICIENTS IN A KNOWN IDENTITY. The first method consists of equating coefficients in a known identity. Thus we may select from the relations existing between the twenty four solutions of the hypergeometric equation,

$$(1-x)^{a+b-c}\ {}_2F_1\left[\begin{matrix}a;\ b\\ c\end{matrix};x\right] = {}_2F_1\left[\begin{matrix}c-a,\ c-b\\ c\end{matrix};x\right].$$

Equating the coefficients of x^n on each side we obtain

$$\sum_{r=0}^{n} \frac{(a)_r\,(b)_r}{r!\,(c)_r}\cdot\frac{(c-a-b)_{n-r}}{(n-r)!} = \frac{(c-a)_n\,(c-b)_n}{n!\,(c)_n}.$$

Hence $\displaystyle\sum_{r=0}^{n} \frac{(a)_r(b)_r(c-a-b)_n(-n)_r}{r!\,(c)_r(1+a+b-c-n)_r\,n!} = \frac{(c-a)_n(c-b)_n}{n!\,(c)_n}$, which is Saalschütz theorem ${}_3F_2\left[\begin{matrix}a,\ b,\ -n\\ c,\ 1+a+b-c-n\end{matrix};\right] = \dfrac{(c-a)_n(c-b)_n}{(c)_n\,(c-a-b)_n}.$

1.7. REARRANGEMENT IN THE ORDER OF SUMMATION OF A DOUBLE SERIES.

The second method consists in rearranging the terms of a double series obtained by replacing certain of the terms of a simple series by the hypergeometric series of which these terms are the reduced form. When the series are non-terminating the question of absolute convergence must be considered to justify the validity of the process. The conditions necessary for absolute convergence will in general impose some restriction on the parameters, a restriction which may possibly by removable by an appeal to the principle of analytic continuation. In the case of terminating series Bailey[1] has obtained a general formula to cover most of the known examples of such transformations. To illustrate the method we insert here a second proof of Saalschütz' theorem, and as an example with non-terminating series we add a proof of the first of the Thomae relations 1.4.(6). reproduced from Bailey's tract.[2]

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[1] tract. p. 23.

[2] tract. p. 14.

In the proof of Saalschütz' theorem we start with Van der Monde's theorem, the same theorem being used in the analysis.

$$\frac{(c-a)_n}{(c)_n} = \sum_{r=0}^{n} \frac{(a)_r(-n)_r}{r!\,(c)_r},$$

$$= \sum_{r=0}^{n} \frac{(a)_r(-n)_r}{r!\,(c-b)_r} \sum_{t=0}^{r} \frac{(b)_t(-r)_t}{t!\,(c)_t},$$

$$= \sum_{t=0}^{n} \frac{(a)_t(b)_t(-n)_t(-)^t}{t!\,(c)_t\,(c-b)_t} \sum_{r=t}^{n} \frac{(a+t)_{r-t}(-n+t)_{r-t}}{(r-t)!\,(c-b+t)_{r-t}},$$

$$= \sum_{t=0}^{n} \frac{(a)_t(b)_t(-n)_t(c-a-b)_{n-t}(-)^t}{t!\,(c)_t\,(c-b)_n},$$

$$= \frac{(c-a-b)_n}{(c-b)_n}\, {}_3F_2\left[\begin{matrix} a,\ b,\ -n \\ c,\ 1+a+b-c-n \end{matrix};\right],$$

and the theorem is proved.

In the proof of the Thomae relation we use Gauss' theorem in the analysis.

$$\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(b_1)\Gamma(b_2)}\, {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right],$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\Gamma(a_2+n)\Gamma(a_3+n)}{n!\,\Gamma(b_1+n)\Gamma(b_2+n)},$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a_2+n)\Gamma(a_3+n)}{n!\,\Gamma(b_1+b_2-a_1+n)}\, {}_2F_1\left[\begin{matrix} b_1-a_1,\ b_2-a_1 \\ b_1+b_2-a_1+n \end{matrix};\right],$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a_2+n)\Gamma(a_3+n)\,\Gamma(b_1-a_1+m)\Gamma(b_2-a_1+m)}{n!\,m!\,\Gamma(b_1+b_2-a_1+n+m)\Gamma(b_1-a_1)\Gamma(b_2-a_1)},$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(b_1-a_1+m)\Gamma(b_2-a_1+m)\Gamma(a_2)\Gamma(a_3)}{m!\,\Gamma(b_1+b_2-a_1+m)\Gamma(b_1-a_1)\Gamma(b_2-a_1)}\, {}_2F_1\left[\begin{matrix} a_2,\ a_3 \\ b_1+b_2-a_1+m \end{matrix};\right],$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(b_1-a_1+m)\Gamma(b_2-a_1+m)\Gamma(a_2)\Gamma(a_3)\Gamma(b_1+b_2-a_1-a_2-a_3+m)}{m!\,\Gamma(b_1+b_2-a_1-a_2+m)\Gamma(b_1+b_2-a_1-a_3+m)\Gamma(b_1-a_1)\Gamma(b_2-a_1)}$$

$$= \frac{\Gamma(a_2)\Gamma(a_3)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_1+b_2-a_1-a_2)\Gamma(b_1+b_2-a_1-a_3)}\, {}_3F_2\left[\begin{matrix} b_1-a_1,\ b_2-a_1,\ b_1+b_2-a_1-a_2-a_3 \\ b_1+b_2-a_1-a_2,\ b_1+b_2-a_1-a_3 \end{matrix};\right]$$

The original ${}_3F_2$ is convergent if $R(b_1+b_2-a_1-a_2-a_3)>0$, and the final ${}_3F_2$ converges if $R(a_1)>0$. The argument also requires that the double series should be absolutely convergent. This is certainly so if the real part of $(b_1+b_2-a_1)$ is sufficiently large. Suppose for example that $R(b_1+b_2-a_1) > 2r+1$, where r is an integer. Then $\Gamma(b_1+b_2-a_1+m+n) > \Gamma(m+r+1)\,\Gamma(n+r+1)$ and the double series may be compared with the product of two absolutely convergent simple series.

1.8. TRANSFORMATIONS USING BARNES' CONTOUR INTEGRALS.

There are two methods by which transformations of hypergeometric series may be obtained using contour integrals of the Barnes' type. In the first the integral is evaluated by summing the residues at poles to the left and right of a contour parallel to the imaginary axis curved if necessary to separate increasing and decreasing sequences of poles. Provided that it can be shown that the integral taken round a large semicircle to the left or right of the imaginary axis tends to zero, then the two values obtained for the integral can be equated. Alternatively it is sufficient simply to consider the integral taken round a large circle.

Consider for example the integral

$$\frac{1}{2\pi i}\int \frac{\Gamma(-s)\,\Gamma(a_2)\,\Gamma(a_3)\,ds}{\Gamma(1-a_1-s)\,\Gamma(b_1+s)\,\Gamma(b_2+s)}$$ taken round a large circle

which avoids the poles of $\Gamma(-s), \Gamma(a_2), \Gamma(a_3)$, i.e. does not pass through the points $n, -a_2-n, -a_3-n$, for any integral n. The integral tends to zero provided that $R(b_1+b_2-a_1-a_2-a_3) > 0$,[1] and since

[1] cf. Whittaker and Watson 'Modern Analysis' 4th ed. p. 289.

the residue due to $\Gamma(-s)$ where $s = n$ is $\frac{(-)^n}{n!}$ [1], we obtain immediately by summing residues at the three sequences of poles,

$$\frac{\Gamma(a_2)\Gamma(a_3)}{\Gamma(1-a_1)\Gamma(b_1)\Gamma(b_2)}\ {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right]$$

$$= \frac{\Gamma(a_2)\Gamma(a_3-a_2)}{\Gamma(1+a_2-a_1)\Gamma(b_1-a_2)\Gamma(b_2-a_2)}\ {}_3F_2\left[\begin{matrix} a_2, 1+a_2-b_1, 1+a_2-b_2 \\ 1+a_2-a_1, 1+a_2-a_3 \end{matrix};\right]$$

+ a similar expression with a_2 and a_3 interchanged.

This is the second of the Thomae relations 1.4.(7).

The second method of obtaining transformations using contour integrals is effectively that used by Barnes[3] to prove the theorem known as Barnes' second lemma, a method which has been considerably extended by Bailey.[4] It consists of replacing certain of the gamma functions occuring in the general term of a hypergeometric series by an equivalent contour integral, interchanging the order of integration and summation and reevaluating the resulting integral. We illustrate the method by the following example extracted from Bailey's tract.[5]

By Barnes' lemma

$$\frac{1}{2\pi i}\int \Gamma(a_1+s)\Gamma(a_2+s)\Gamma(n-s)\Gamma(b_1-a_1-a_2-s)\,ds = \frac{\Gamma(a_1+n)\Gamma(a_2+n)\Gamma(b_1-a_1)\Gamma(b_1-a_2)}{\Gamma(b_1+n)}.$$

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(1) tract. p. 6. (2) Whipple has used this method extensively to obtain relations between well-poised ${}_7F_6$. Whipple. 3.

(3) Barnes. 2. The proof is reproduced in Bailey's tract pp. 42-43.

(4) Bailey. 1. vide tract chapter VI. (5) tract. p. 42.

Thus $${}_3F_2\left[\begin{matrix}a_1, a_2, a_3\,;\\ b_1, b_2\,;\end{matrix}\right] = \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)\Gamma(a_1)\Gamma(a_2)}\,.$$
$$\cdot\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int\frac{(a_3)_n}{n!\,(b_2)_n}\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(n-s)\Gamma(b_1-a_1-a_2-s)\,ds,$$
$$= \frac{\Gamma(b_1)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)\Gamma(a_1)\Gamma(a_2)}\,.$$
$$\frac{1}{2\pi i}\int\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(b_1-a_1-a_2-s)\Gamma(-s)\,{}_2F_1\left[\begin{matrix}a_3, -s\,;\\ b_2\end{matrix}\right]ds,$$
$$= \frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)\Gamma(b_2-a_3)\Gamma(a_1)\Gamma(a_2)}\,.$$
$$\frac{1}{2\pi i}\int\frac{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(b_1-a_1-a_2-s)\Gamma(b_2-a_3+s)\Gamma(-s)}{\Gamma(b_2+s)}\,ds\,.$$

The interchange in the order of summation and integration is justified if $R(b_2-a_3+s)>0$. If we now evaluate the integral by considering residues due to the poles of $\Gamma(b_1-a_1-a_2-s)$, $\Gamma(-s)$, we obtain one of the Thomae three term relations,
$${}_3F_2\left[\begin{matrix}a_1, a_2, a_3\,;\\ b_1, b_2\,;\end{matrix}\right] = \frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)}\,{}_3F_2\left[\begin{matrix}a_1, a_2, b_2-a_3\,;\\ b_2, 1+a_1+a_2-b_1\,;\end{matrix}\right]$$
$$+\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)\Gamma(a_1+a_2-b_1)}{\Gamma(b_2-a_3)\Gamma(a_1)\Gamma(a_2)\Gamma(b_1+b_2-a_1-a_2)}$$
$${}_3F_2\left[\begin{matrix}b_1-a_2, b_1-a_1, b_1+b_2-a_1-a_2-a_3\,;\\ b_1+b_2-a_1-a_2, 1+b_1-a_1-a_2\,;\end{matrix}\right].$$

1.9. DOUGALL'S METHOD. When a transformation of a terminating series has been discovered, or has been inferred empirically, it may often be proved very elegantly by a method due to Dougall.[1] Thus we may prove Saalschütz' theorem in the following manner.

Suppose
$$(1)\quad {}_3F_2\left[\begin{matrix}a_1, a_2, a_3\,;\\ b_1, b_2\,;\end{matrix}\right] = \frac{\Gamma(b_1-a_1-a_3)\Gamma(b_1-a_2-a_3)\Gamma(b_1-a_1-a_2)\Gamma(b_1)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)\Gamma(b_1-a_3)\Gamma(b_1-a_1-a_2-a_3)},$$

where $b_1+b_2-a_1-a_2-a_3=1$,

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[1] Dougall. 1. vide tract p. 34.

is true when $-a_3 = 0, 1, \cdots (m-1)$. Then by symmetry it is also true when $-a_1$, or $-a_2 = 0, 1, \cdots (m-1)$. It is therefore true in the particular case where $a_3 = -m$ for $2m$ values of a_2, namely

$$a_2 = 0, -1, \cdots, -(m-1) \quad ; \text{ and } a_2 = b_1 + b_2 - a_3 - 1 + r, \; r = 0 \text{ to } (m-1).$$

Setting $a_3 = -m$, equation (1) becomes

$$ {}_3F_2\left[\begin{matrix} b_1+b_2-a_2+m-1, a_2, -m ; \\ b_1, b_2 \end{matrix}\right] = \frac{(1+a_2-b_2-m)_m (b_1-a_2)_m}{(b_1)_m \; (1-b_2-m)_m} \tag{2}$$

which is a relation between two polynomials of degree $2m$ in a_2. We know that this equation is satisfied for $2m$ values of a_2. If we can show that it is satisfied for one further value of a_2 then it becomes an identity, and since (1) is easily seen to be true for $a_3 = 0$, we have proved the theorem by induction. To complete the proof therefore, it is only necessary to set $a_2 = b_2$ when the Saalschützian condition becomes $1 + a_1 - b_1 = m$ and both sides of (2) vanish separately.

1.10. TRANSFORMATIONS OBTAINED USING DEFINITE INTEGRALS.

Certain of the Thomae relations connecting two or three series of the type ${}_3F_2$ may be shown to depend on the relations between two or three of the twenty four solutions of the hypergeometric equation. The method involves an interchange in the order of summation and integration of an infinite series, and to ensure the validity of the process some restriction must in general be placed on the parameters. But once again this restriction may be removed by an appeal to the principle of analytic continuation, and the final result will be true for those values of the parameters which make all the series concerned convergent. An example of the method occurs in a paper by Hardy.[1] As an illustration we will use the method to obtain the Thomae relation 1.4.(6).

$$ {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right] = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)} \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \Gamma(a_3+n)\Gamma(b_2-a_3)}{n!(b_1)_n \Gamma(b_2+n)}, $$

$$ = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)} \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{n!\,(b_1)_n} \int_0^1 t^{a_3+n-1}(1-t)^{b_2-a_3-1}\,dt, $$

and formally,

$$ (1)\quad {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right] = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)} \int_0^1 t^{a_3-1}(1-t)^{b_2-a_3-1} {}_2F_1\left[\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; t\right] dt. $$

If we replace the ${}_2F_1$ in the integrand using the relation

$$ {}_2F_1\left[\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; t\right] = (1-t)^{b_1-a_1-a_2} {}_2F_1\left[\begin{matrix} b_1-a_1, b_1-a_2 \\ b_1 \end{matrix}; t\right] $$

then the right hand side of (1) becomes

$$ \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)} \int_0^1 \sum_{n=0}^{\infty} t^{a_3+n-1}(1-t)^{b_1+b_2-a_1-a_2-a_3-1} \frac{(b_1-a_1)_n (b_1-a_2)_n}{n!\,(b_1)_n}\,dt, $$

$$ = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2-a_3)} \sum_{n=0}^{\infty} \frac{(b_1-a_1)_n (b_1-a_2)_n}{n!\,(b_1)_n} \cdot \frac{\Gamma(a_3+n)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_1+b_2-a_1-a_2+n)}, $$

so that eventually we have

$$ {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix};\right] = \frac{\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_1+b_2-a_1-a_2)\Gamma(b_2-a_3)} {}_3F_2\left[\begin{matrix} b_1-a_1, b_1-a_2, a_3 \\ b_1, b_1+b_2-a_1-a_2 \end{matrix};\right]. $$

A repitition of this transformation gives the required Thomae form 1.4.(6).

A slight variant of this method gives rise to the following interesting identity due to Whipple,

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[1] Hardy 1. pp. 498-499.

Let $F(a,b,c) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_4F_3\left[\begin{matrix} a,\ b,\ \tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2} \\ \tfrac{1}{2}(a+b),\ \tfrac{1}{2}(a+b+1),\ c+1 \end{matrix};\right]$,

(2) Then $F(a,b,c) + F(a-c,b-c,-c) = \dfrac{\Gamma(a)\Gamma(b-c)+\Gamma(b)\Gamma(a-c)}{\Gamma(a+b-c)}$ [1].

This formula is a generalisation for non-terminating series of Whipple (11.3), and is obtained by setting $x=0$ in

$$\frac{1}{\Gamma(a+x-c)\Gamma(2x-c)\Gamma(1+c-x)}\left[\frac{1}{\Gamma(b-c)\Gamma(a+b)}\,{}_4F_3\left[\begin{matrix} a,\ b-x,\ \tfrac{1}{2}c,\ \tfrac{1}{2}c+\tfrac{1}{2} \\ 1+c-x,\ \tfrac{1}{2}(a+b),\ \tfrac{1}{2}(a+b+1) \end{matrix};\right] - \frac{1}{\Gamma(a+b-c)\Gamma(b)}\,{}_3F_2\left[\begin{matrix} c,\ x,\ a \\ 1+c-x,\ b \end{matrix};\right]\right]$$

$$= \frac{1}{\Gamma(a)\Gamma(c)\Gamma(1+x-c)}\left[\frac{1}{\Gamma(a+b+2x-2c)\Gamma(b-x)}\,{}_4F_3\left[\begin{matrix} a-c+x,\ b-c,\ x-\tfrac{1}{2}c,\ x-\tfrac{1}{2}c+\tfrac{1}{2} \\ 1-c+x,\ \tfrac{1}{2}(a+b)-c+x,\ \tfrac{1}{2}(a+b+1)-c+x \end{matrix};\right] - \frac{1}{\Gamma(a+b-c)\Gamma(b+x-c)}\,{}_3F_2\left[\begin{matrix} 2x-c,\ x,\ a-c+x \\ 1+x-c,\ b+x-c \end{matrix};\right]\right],$$

which is also due to Whipple [2]

The proof of (2) given here is due to Burchnall, and makes use of the Gauss transformation [3]

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ a+b+\tfrac{1}{2} \end{matrix};\ 4t(1-t)\right] = {}_2F_1\left[\begin{matrix} 2a,\ 2b \\ a+b+\tfrac{1}{2} \end{matrix};\ t\right], \quad 0 \le t \le \tfrac{1}{2}$$

$$= {}_2F_1\left[\begin{matrix} 2a,\ 2b \\ a+b+\tfrac{1}{2} \end{matrix};\ 1-t\right], \quad \tfrac{1}{2} \le t \le 1,$$

which becomes in the particular case $b = a+\tfrac{1}{2}$

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(1) I am indebted to Dr Chaundy for showing me this identity which was communicated to him by Whipple shortly before his death, and which was never published.

(2) This formula is deducible from equations 4.4.(4)., 6.5.(1), and 3.2.(2) of Bailey's tract.

(3) tract. 10.2.(2).

$$(3)\qquad {}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2} \\ 2a+1 \end{matrix};4t(1-t)\right] = (1-t)^{-2a}, \qquad 0 \leq t \leq \tfrac{1}{2},$$
$$= t^{-2a}, \qquad \tfrac{1}{2} \leq t \leq 1.$$

$$\begin{aligned}
F(a,b,c) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\, {}_4F_3\left[\begin{matrix} a,\ b,\ \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2} \\ \frac{1}{2}(a+b),\ \frac{1}{2}(a+b+1),\ c+1 \end{matrix};\right], \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}c)_n(\frac{1}{2}c+\frac{1}{2})_n}{n!\,(c+1)_n}\, 2^{2n}\, \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a+b+2n)}, \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}c)_n(\frac{1}{2}c+\frac{1}{2})_n}{n!\,(c+1)_n}\, 2^{2n} \int_0^1 t^{a+n-1}(1-t)^{b+n-1}\,dt, \\
&= \int_0^{\frac{1}{2}} t^{a-1}(1-t)^{b-1}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2} \\ c+1 \end{matrix};4t(1-t)\right]dt \\
&\quad + \int_0^{\frac{1}{2}} t^{b-1}(1-t)^{a-1}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}c,\ \frac{1}{2}c+\frac{1}{2} \\ c+1 \end{matrix};4t(1-t)\right]dt, \\
&= \int_0^{\frac{1}{2}} t^{a-1}(1-t)^{b-c-1}\,dt + \int_0^{\frac{1}{2}} t^{b-1}(1-t)^{a-c-1}\,dt, \\
&= 2^{-a}\int_0^1 u^{a-1}(1-\tfrac{1}{2}u)^{b-c-1}\,du + 2^{-b}\int_0^1 u^{b-1}(1-\tfrac{1}{2}u)^{a-c-1}\,du,
\end{aligned}$$

on setting $t = \frac{1}{2}u$,

$$= (a2^a)^{-1}\, {}_2F_1\left[\begin{matrix} a,\ 1+c-b \\ a+1 \end{matrix};\tfrac{1}{2}\right] + (b2^b)^{-1}\, {}_2F_1\left[\begin{matrix} b,\ 1+c-a \\ b+1 \end{matrix};\tfrac{1}{2}\right].$$

We now transform the series on the right hand side using

$$\begin{aligned}
{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};x\right] &= (1-x)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a,\ c-b \\ c \end{matrix};x\right] \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1\left[\begin{matrix} a,\ b \\ 1+a+b-c \end{matrix};1-x\right] \\
&\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a,\ c-b \\ 1+c-a-b \end{matrix};1-x\right]
\end{aligned}$$

and obtain

$$(4)\qquad F(a,b,c) = 2^{c-a-b}\left\{\frac{1}{a}\, {}_2F_1\left[\begin{matrix} 1,\ a+b-c \\ 1+a \end{matrix};\tfrac{1}{2}\right] + \frac{1}{b}\, {}_2F_1\left[\begin{matrix} 1,\ a+b-c \\ 1+b \end{matrix};\tfrac{1}{2}\right]\right\}$$

$$\begin{aligned}
(5)\qquad F(a,b,c) &= \frac{\Gamma(a)\Gamma(b-c)+\Gamma(b)\Gamma(a-c)}{\Gamma(a+b-c)} \\
&\quad + 2^{c-a-b}\left\{\frac{1}{c-b}\, {}_2F_1\left[\begin{matrix} 1,\ a+b-c \\ 1+b-c \end{matrix};\tfrac{1}{2}\right] + \frac{1}{c-a}\, {}_2F_1\left[\begin{matrix} 1,\ a+b-c \\ 1+a-c \end{matrix};\tfrac{1}{2}\right]\right\}.
\end{aligned}$$

By (4) we see that

$$(6)\quad F(a-c, b-c, -c) = 2^{c-a-b}\left\{\frac{1}{a-c}\,{}_2F_1\left[\begin{matrix}1, a+b-c\\ 1+a-c\end{matrix}; \tfrac{1}{2}\right] + \frac{1}{b-c}\,{}_2F_1\left[\begin{matrix}1, a+b-c\\ 1+b-c\end{matrix}; \tfrac{1}{2}\right]\right\},$$

and adding (5) and (6) we obtain the required result

$$F(a, b, c) + F(a-c, b-c, -c) = \frac{\Gamma(a)\Gamma(b-c) + \Gamma(b)\Gamma(a-c)}{\Gamma(a+b-c)}.$$

1.11. RECIPROCAL THEOREMS. Burchnall and Chaundy[1] have obtained a number of transformations of Appell's series into extended hypergeometric series, transformations which group themselves naturally into pairs. Although these pairs of relations were independently obtained, their interdependence follows from a number of theorems of a reciprocal nature which have been noticed by Burchnall and form the subject of a subscript to a later paper by Chaundy.[2] We collect these theorems here, together with their proofs in the cases where the series concerned terminate, and show how the results can be used to obtain transformations of terminating series. Whilst it is not difficult to formulate conditions under which the theorems remain true for non-terminating series, the problem of obtaining best possible conditions proves to be somewhat intractable. Rinfoot and Shephard[3] have, in the course of a different investigation, considered what is in effect one of these reciprocal theorems, and have obtained sets of sufficient conditions in the infinite case.

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(1) Burchnall and Chaundy. 1. (2) Chaundy. 1.

(3) Rinfoot and Shephard. 1 and 2.

Theorem I. If $\phi_r = \sum_{t=0}^{m-r} \frac{1}{t!}\Theta_{t+r}$, then $\Theta_r = \sum_{t=0}^{m-r} \frac{(-)^t}{t!}\phi_{t+r}$,

and conversely.

It is sufficient to prove the theorem for the case $r = 0$.

In $\sum_{t=0}^{m} \frac{(-)^t}{t!}\phi_t$ the coefficient of Θ_s is $\sum_{t=0}^{s} \frac{(-)^t}{t!(s-t)!}$,

i.e. $\frac{1}{s!}\sum_{t=0}^{s}\frac{(-s)_t}{t!}$,

which is equal to zero except when $s = 0$, the coefficient of Θ_0 being unity. The proof of the converse is similar.

Theorem II. If $\phi_r = \sum_{t=0}^{m-r}\frac{(\lambda)_t}{t!}\Theta_{r+t}$, then $\Theta_r = \sum_{t=0}^{m-r}\frac{(-\lambda)_t}{t!}\phi_{r+t}$,

and conversely.

Again it is sufficient to prove the theorem for $r = 0$.

In $\sum_{t=0}^{m}\frac{(-\lambda)_t}{t!}\phi_t$ the coefficient of Θ_s is $\sum_{t=0}^{s}\frac{(-\lambda)_t(\lambda)_{s-t}}{t!(s-t)!}$,

$$= \frac{(\lambda)_s}{s!}\sum_{t=0}^{s}\frac{(-\lambda)_t(-s)_t}{t!(1-\lambda-s)_t},$$

$$= \frac{(\lambda)_s}{s!}\frac{(1-s)_s}{(1-\lambda-s)_s}$$

$= 0$, $s \neq 0$, and the coefficient of Θ_0 is unity. The converse is proved in a similar manner.

Theorem III. If $\phi_r = \sum_{t=0}^{m-r}\frac{1}{t!(c+2r+t-1)_t}\Theta_{r+t}$,

then $\Theta_r = \sum_{t=0}^{m-r}\frac{(-)^t}{t!(c+2r)_t}\phi_{r+t}$.

Again it suffices to consider $r = 0$.

In $\sum_{t=0}^{m}\frac{(-)^t}{t!(c)_t}\phi_t$ the coefficient of Θ_s is $\sum_{t=0}^{s}\frac{(-)^t}{t!(c)_t(s-t)!(c+t+s-1)_{s-t}}$

$$= \frac{1}{s!(c+s-1)_s}\sum_{t=0}^{s}\frac{(-s)_t(c+s-1)_t}{t!(c)_t}$$

$$= \frac{1}{s!(c+s-1)_s}\cdot\frac{(1-s)_s}{(c)_s} = 0, \quad s \neq 0,$$

and the coefficient of Θ_0 is unity.

To prove the converse consider the coefficient of ϕ_s in $\sum_{t=0}^{m} \frac{1}{(c+t-1)_t\, t!}\Theta_t$.

This is $\displaystyle\sum_{t=0}^{s} \frac{(-)^{s-t}}{(c+t-1)_t\, t!\,(s-t)!\,(c+2t)_{s-t}}$,

$$= \frac{(-)^s}{s!\,(c)_s}\sum_{t=0}^{s}\frac{(c+2t-1)\,(c)_{t-1}\,(-s)_t}{t!\,(c+s)_t}\,,$$ in which the general term

$$\frac{(-s)_t\,(c)_{t-1}}{t!\,(c+s)_t}(c+2t-1) = \frac{(1-s)_t\,(c)_t}{t!\,(c+s)_t} - \frac{(1-s)_{t-1}\,(c)_{t-1}}{(t-1)!\,(c+s)_{t-1}}\,,$$

and therefore the series vanishes. Also the coefficient of ϕ_0 is equal to unity which completes the proof of the converse.

As an illustration of how these theorems may be used to obtain transformations of series we prove Whipple's relation connecting a 'well-poised' ${}_7F_6$ and a Saalschützian ${}_4F_3$ for the case where both series terminate, using only Saalschütz' theorem in the analysis. By Saalschütz we have

$$\frac{(a_1-a_4)_m(a_1-a_5)_m}{(a_1)_m(a_1-a_4-a_5)_m} = \sum_{n=0}^{m}\frac{(a_4)_n(a_5)_n(-m)_n}{n!\,(a_1)_n(1+a_4+a_5-a_1-m)_n}\,,$$

$$= \sum_{n=0}^{m}\frac{(a_4)_n(a_5)_n(a_1-a_2-a_3)_n(-m)_n}{n!\,(a_1-a_2)_n(a_1-a_3)_n(1+a_4+a_5-a_1-m)_n}\cdot\frac{(a_1-a_2)_n(a_1-a_3)_n}{(a_1)_n(a_1-a_2-a_3)_n}\,,$$

$$= \sum_{n=0}^{m}\frac{(a_4)_n(a_5)_n(a_1-a_2-a_3)_n(-m)_n}{n!\,(a_1-a_2)_n(a_1-a_3)_n(1+a_4+a_5-a_1-m)_n}\cdot\sum_{r=0}^{n}\frac{(a_2)_r(a_3)_r(-n)_r}{r!\,(a_1)_r(1+a_2+a_3-a_1-n)_r}\,,$$

$$= \sum_{r=0}^{m}\sum_{n=r}^{m}\frac{(a_2)_r(a_3)_r(a_4)_r(a_5)_r(-m)_r}{r!\,(a_1)_r(a_1-a_2)_r(a_1-a_3)_r(1+a_4+a_5-a_1-m)_r}\,.$$

$$\cdot\frac{(a_1-a_2-a_3)_{n-r}(a_4+r)_{n-r}(-m+r)_{n-r}(a_5+r)_{n-r}}{(n-r)!\,(a_1-a_2+r)_{n-r}(a_1-a_3+r)_{n-r}(1+a_4+a_5-a_1-m+r)_{n-r}}\,,$$

and therefore

$$(1)\qquad \frac{(a_1-a_4)_m(a_1-a_5)_m}{(a_1)_m(a_1-a_4-a_5)_m} = \sum_{r=0}^{m}\frac{(a_2)_r(a_3)_r(a_4)_r(a_5)_r(-m)_r}{r!\,(a_1)_r(a_1-a_2)_r(a_1-a_3)_r(1+a_4+a_5-a_1-m)_r}$$

$${}_4F_3\left[\begin{matrix}a_1-a_2-a_3,\ a_4+r,\ a_5+r,\ -m+r\\ a_1-a_2+r,\ a_1-a_3+r,\ 1+a_4+a_5-a_1-m+r\end{matrix};\right]$$

In Theorem III let

$$\phi_r = \frac{(-)^r (a_2)_r (a_3)_r (a_4)_r (a_5)_r (-m)_r}{(a_1-a_2)_r (a_1-a_3)_r (1+a_4+a_5-a_1-m)_r} {}_4F_3\left[\begin{matrix} a_1-a_2-a_3,\ a_4+r,\ a_5+r,\ -m+r \\ a_1-a_2+r,\ a_1-a_3+r,\ 1+a_4+a_5-a_1-m+r \end{matrix};\right]$$

then $\Theta_r = \sum_{t=0}^{m-r} \frac{(-)^t}{t!\,(a_1+2r)_t} \phi_{r+t}$, setting $c = a_1$

$$= (-)^r \sum_{t=0}^{m-r} \frac{(a_2)_{r+t} (a_3)_{r+t} (a_4)_{r+t} (a_5)_{r+t} (-m)_{r+t}}{t!\,(a_1+2r)_t (a_1-a_2)_{r+t} (a_1-a_3)_{r+t} (1+a_4+a_5-a_1-m)_{r+t}} \cdot$$

$$\cdot {}_4F_3\left[\begin{matrix} a_1-a_2-a_3,\ a_4+r+t,\ a_5+r+t,\ -m+r+t \\ a_1-a_2+r+t,\ a_1-a_3+r+t,\ 1+a_4+a_5-a_1-m+r+t \end{matrix};\right],$$

$$= (-)^r \frac{(a_2)_r (a_3)_r (a_4)_r (a_5)_r (-m)_r}{(a_1-a_2)_r (a_1-a_3)_r (1+a_4+a_5-a_1-m)_r} \cdot \frac{(a_1-a_4+r)_{m-r}\ (a_1-a_5+r)_{m-r}}{(a_1+2r)_{m-r}\ (a_1-a_4-a_5)_{m-r}}$$

by equation (1).

Therefore $\Theta_r = \frac{(a_1-a_4)_m (a_1-a_5)_m}{(a_1)_m (a_1-a_4-a_5)_m} \cdot \frac{(a_1)_{2r} (a_2)_r (a_3)_r (a_4)_r (a_5)_r (-m)_r}{(a_1-a_2)_r (a_1-a_3)_r (a_1-a_4)_r (a_1-a_5)_r (a_1+m)_r}$,

and the relation $\phi_0 = \sum_{r=0}^{m} \frac{1}{r!\,(a_1+r-1)_r} \Theta_r$ gives

$$ {}_4F_3\left[\begin{matrix} a_1-a_2-a_3,\ a_4,\ a_5,\ -m \\ a_1-a_2,\ a_1-a_3,\ 1+a_4+a_5-a_1-m \end{matrix};\right]$$

$$= \frac{(a_1-a_4)_m (a_1-a_5)_m}{(a_1)_m (a_1-a_4-a_5)_m} {}_7F_6\left[\begin{matrix} a_1-1,\ \tfrac{1}{2}a_1+\tfrac{1}{2},\ a_2,\ a_3,\ a_4,\ a_5,\ -m \\ \tfrac{1}{2}a_1-\tfrac{1}{2},\ a_1-a_2,\ a_1-a_3,\ a_1-a_4,\ a_1-a_5,\ a_1+m \end{matrix};\right]$$

which is Whipples relation.

1.12. SCOPE OF THE FOLLOWING CHAPTERS.

In hypergeometric theory the operator $\delta \equiv x\frac{d}{dx}$ is a much more natural tool to employ than the ordinary operation of differentiation $\frac{d}{dx}$. Thus for example the differential equation satisfied by the function $F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right]$ when thrown into 'δ-form' takes the particularly simple form

$$\left\{\delta \prod(\delta+b_s-1) - x \prod(\delta+a_r)\right\} y = 0,$$

mere inspection of which enables us to write down other solutions, or detect whether the equation has a polynomial solution, etc.

points which are not immediately obvious when the equation is written in the more usual form $a_0\frac{d^ny}{dx^n} + \cdots + a_ny = 0$.

This thesis consists of a series of investigations into the power of 'δ-methods' in various branches of hypergeometric theory. In this chapter we have listed the most important transformations of series with fixed argument and given a very brief account of the methods by which they have been obtained. In the next chapter a technique is developed, depending on 'δ-methods' which gives most of the known identities concerning reducible series in a simple manner. In chapter III, using a basic operator analogous to δ, a similar technique is developed for basic series. This chapter also contains a resumé of some of the work of F. H. Jackson on basic series. In the fourth chapter we consider the system of partial differential equations satisfied by Appell's function $F_{(1)}$. Sixty solutions of these equations in the form of functions of the type $F_{(1)}$ are listed by Appell and Kampé de Fériet,[1] and other solutions are known in terms of Appell's functions $F_{(2)}$ and $F_{(3)}$, and also in terms of Horn's functions $G_{(2)}$ and $H_{(2)}$.[2] By transforming the equations themselves, changing one or both of the variables, all these solutions are readily and systematically

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(1) 'Appell and Kampé de Fériet' pp. 62-64.

(2) Erdélyi. 1.

obtained. In chapter V we consider the differential equation satisfied by the general $_3F_2$ and obtain solutions in the neighbourhood of the singularity at $x = 1$. This problem has been treated by Darling.[1] Here using the 'δ-form' of the equation we are able to obtain Darling's solutions in what appears to be a more natural manner.

[1] Darling. I.

Chapter II

Reducible series. Transformations of series. An operational method.

2.1. Reducible series. If $F(a)$ is a reducible series with some fixed argument then by definition

$$F(a) = A \frac{\prod\{\Gamma(a+b_r)\}}{\prod\{\Gamma(a+c_r)\}}$$

where A is independent of a. The contiguous series

$$F(a+1) = A \frac{\prod\{\Gamma(a+b_r+1)\}}{\prod\{\Gamma(a+c_r+1)\}} = \frac{\prod(a+b_r)}{\prod(a+c_r)} F(a).$$

A reducible series regarded as a function of one of its parameters satisfies therefore a recurrence relation of the form

$$\prod(a+c_r)\, F(a+1) = \prod(a+b_r)\, F(a).$$

In order to ensure that a series of order greater than two satisfies a two term recurrence relation it is in general necessary to impose conditions on the parameters and these conditions may be disturbed when we replace a by $a+1$. If however the conditions are preserved when we replace $a+r$ by $a+r+1$ for all r, then we may proceed to

$$F(a+n) = \frac{\prod(a+b_r)_n}{\prod(a+c_r)_n} F(a),$$

and the series is reducible, subject to the imposed conditions, if either

$$a = -n,$$

so that the initial series terminates and $F(a+n)$ is equal to unity, or $\lim_{n\to\infty} F(a+n)$ is itself some product of gamma functions.

In either case a first step in the argument is to obtain the recurrence relation satisfied by $F(a)$ and to do this we employ an operational method outlined below. A feature of the approach is that it indicates the type of series which may be expected to be reducible and the special position of 'well-poised' and terminating 'Saalschützian' series no longer appears as an arithmetical accident but follows naturally from the analysis.

2.2. METHOD OF OBTAINING RECURRENCE RELATIONS.[1]

We have seen in chapter I that if $F\left[\begin{matrix} a_r \\ b_s \end{matrix}; x\right]$ is denoted by F then

(1) $$\delta\, \Pi(b_s)\, F = x\, \Pi(a_r)\, F\left[\begin{matrix} a_r+1 \\ b_s+1 \end{matrix}; x\right],$$

(2) $$(\delta + a_r)\, F = a_r F\left[\begin{matrix} a_r+1 \\ b_s \end{matrix}; x\right],$$

(3) $$(\delta + b_s - 1) F = (b_s - 1) F\left[\begin{matrix} a_r \\ b_s - 1 \end{matrix}; x\right],$$

where in (1) all the parameters are augmented by unity on the right hand side, whilst in (2) and (3) only the parameter specialised is augmented or diminished.

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[1] An outline of this method appears in Burchnall and Lakin 1., where reference is made to the fact that the method has been used in another connection by Chaundy, vide. Chaundy 2.

To these we add

(4) $$(\delta+c)_{p}F_{p-1} = c\ {}_{p+1}F_{p}\left[\begin{matrix} a_r, & c+1 \\ b_s; & c \end{matrix};x\right],$$

where c does not belong to a_r.

From (1) - (3) we also deduced the differential equation satisfied by F, namely

(5) $$\delta\,\Pi(\delta+b_s-1)\,F = x\,\Pi(\delta+a_r)\,F.$$

If now in (1) - (4) we set $x=1$ the operation δ becomes meaningless, but δ, $\delta+a_r$, $\delta+b_s-1$, etc. may now be regarded as operating on the parameters a_r, b_s with effects defined by the above identities. Alternatively we may suppose that x is set equal to unity after the differentiations have been performed. We thus see from (5) that the series with argument unity is annihilated by an operator of the form

(6) $$[\delta\,\Pi(\delta+b_s-1) - \Pi(\delta+a_r)].$$

We now seek to write this operator as

(7) $$[A f(\delta) - B g(\delta)] \qquad \text{where}$$

$$A f(\delta) F = A' F'(a+1); \quad B g(\delta) F = B' F(a),$$

so that performing the operations implied by (7) we obtain

$$A' F'(a+1) = B' F'(a)$$

a recurrence relation of the required form satisfied by $F'(a)$.

It is very important to observe however, that whereas when $|x|<1$ the series $F\left[\begin{matrix} a_r \\ b_s \end{matrix};x\right]$ may be differentiated as often as we please, the series $F\left[\begin{matrix} a_r \\ b_s \end{matrix};\right]$ only converges if the real part of its 'index of convergence' $\sum_1 b_s - \sum_1 a_r$ is greater than zero, and that this index is diminished by unity by each δ-operation of the type considered. If in order to obtain the form (7) from (6) no restriction is placed on the values of a_r, b_s, we may provisionally assume that these values are such as to make all the series considered convergent. Any theorem deduced from (7) will then be proved for a restricted range of parameter values, and if the theorem is significant for a wide range of values, the restriction may be removed by analytic continuation. If however to obtain a suitable form for (7) we have to impose some condition on the parameters then this condition may be inconsistent with the convergence of some or all of the series concerned. In these circumstances results may still be deduced for terminating series. The method may also be applied 'mutatis mutandis' to series of argument -1.

2.3. CARLSON'S THEOREM.[1] As a final process we may sometimes make use of the following theorem due to Carlson.

Theorem. $f(z)$ is an analytical function of the complex variable z which satisfies the following three conditions,

(i) $f(z)$ is regular for $R(z) \geq 0$

(ii) $|f(z)| < Ce^{k|z|}$ where $k < \pi$ for $R(z) \geq 0$

(iii) $f(z) = 0$ for $z = 0, 1, 2, \cdots$,

then $f(z) = 0$ identically.

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[1] tract. 5.3.

2.4. GAUSS' THEOREM. As a first illustration of the method and for the sake of completeness we will prove Gauss' Theorem.

Let $F(c+1) = {}_2F_1\left[\begin{matrix} a, b; \\ c+1 \end{matrix}\right]$,

then $[\delta(\delta+c) - (\delta+a)(\delta+b)]\, F(c+1) = 0$,

and we will assume provisionally that $R(c-a-b-1) > 0$, so that all the operations are valid. The operator inside the square bracket is linear in δ, and may be written in the form

$$[(\delta+c)(c-a-b) - (c-a)(c-b)],$$

and therefore by 2.2.(3).

$$c(c-a-b)\, F(c) = (c-a)(c-b)\, F(c+1).$$

The proof now follows that given in Bailey's tract.[1] Repeating the process we see that

$$F(c) = \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m}\, F(c+m).$$

$$\lim_{m\to\infty} \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m} = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.^{(2)}$$

If $v_n(a,b,c)$ is the coefficient of x^n in $F(c)$, and $m > |c|$, we have

$$|F(c+m) - 1| \leqslant \sum_{n=1}^{\infty} |v_n(a,b,c+m)|,$$

$$\leqslant \sum_{n=1}^{\infty} |v_n(|a|,|b|,m-|c|)|,$$

$$< \frac{a.b}{m-|c|} \sum_{n=0}^{\infty} v_n(|a|+1,|b|+1,m+1-|c|).$$

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[1] tract. 1.3.

[2] tract. 1.3.

The last series converges when $m > |c|+|a|+|b|+1$ and is a positive decreasing function of m. Hence

$$\lim_{m\to\infty} F(c+m) = 1,$$

and we have proved Gauss' theorem

$${}_2F_1\left[\begin{matrix} a, b ; \\ c \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

We may now remove the initial restriction under which the recurrence relation was obtained, and the theorem is true provided only that the series converges, i.e. provided $R(c-a-b) > 0$.

2.5. SAALSCHÜTZ' THEOREM.

$F = {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 ; \\ b_1, b_2 \end{matrix}\right]$ satisfies the relation

(1) $$[\delta(\delta+b_1-1)(\delta+b_2-1) - (\delta+a_1)(\delta+a_2)(\delta+a_3)]F = 0.$$

To make the bracketed operator linear in δ, we impose the condition

(2) $$b_1+b_2-a_1-a_2-a_3 = 2,$$

and since this condition is inconsistent with the convergence of $\delta^3 F$ we add the further restriction that a_3 is a negative integer $-n$, so that the series terminates.

We may rewrite the operator in (1) in the form

$$[A(\delta+a_1) - B(\delta+a_3)],$$

where setting $\delta = -a_1$, $\delta = -a_3$ in turn, we obtain

$$B(a_3-a_1) = -a_1(b_1-a_1-1)(b_2-a_1-1),$$

$$A(a_1-a_3) = -a_3(b_1-a_3-1)(b_2-a_3-1).$$

Thus from 2.2.(2) we have

(3) $$(1+a_1-b_1)(1+a_1-b_2)\,{}_3F_2\left[\begin{matrix}a_1, a_2, a_3+1\\ b_1, b_2\end{matrix};\right] = (1+a_3-b_1)(1+a_3-b_2)\,{}_3F_2\left[\begin{matrix}a_1+1, a_2, a_3\\ b_1, b_2\end{matrix};\right].$$

If in (3) we change a_1 into a_1-1, we have in place of (2) the ordinary Saalschützian condition

(4) $$b_1+b_2-a_1-a_2+n = 1,$$

and from (3)

$${}_3F_2\left[\begin{matrix}a_1, a_2, -n\\ b_1, b_2\end{matrix};\right] = \frac{(b_1-a_1)(b_2-a_1)}{(b_1+n-1)(b_2+n-1)}\,{}_3F_2\left[\begin{matrix}a_1-1, a_2, -n+1\\ b_1, b_2\end{matrix};\right].$$

The parameters of the ${}_3F_2$ on the right of this relation still satisfy condition (4), and the process may be repeated, the integral parameter being reduced by unity at each stage. Thus eventually, subject to (4), we have

$${}_3F_2\left[\begin{matrix}a_1, a_2, -n\\ b_1, b_2\end{matrix};\right] = \frac{(b_1-a_1)_n(b_2-a_1)_n}{(b_1)_n(b_2)_n}.$$

2.6. THE SPECIAL POSITION OF THE 'WELL-POISED' SERIES.

The differential equation satisfied by the 'well-poised' series ${}_{s+1}F_s\left[\begin{matrix}2a_1, a_1+a_r\\ 1+a_1-a_r\end{matrix}; x\right]$, takes the form

$$\left[\delta\prod_{r=2}^{s}(\delta+a_1-a_r) - x\prod_{r=2}^{s}(\delta+a_1+a_r)\right]y = 0.$$

Using the property of the 'δ-operator'

$$f(\delta)x^{a_1}y = x^{a_1}f(\delta+a_1)y,$$

we see that $z = x^{a_1}F\left[\begin{matrix}2a_1, a_1+a_r\\ 1+a_1-a_r\end{matrix}; x\right]$ is a solution of the 'well-poised' equation

(1) $$\left[x\prod(\delta+a_r) - \prod(\delta-a_r)\right]z = 0, \qquad r = 1,2,\dots s.$$

The other solutions of this equation are of course

$$x^{a_t} F\left[\begin{matrix}2a_t, a_t+a_r \\ 1+a_t-a_r\end{matrix}; x\right] \quad , r = 1, 2, \cdots t-1, t+1, \cdots s.,$$

and it is also interesting to note that the equation has exactly the same form if we set $\frac{1}{x}$ for x, and therefore that it has solutions valid at infinity

$$x^{-a_t} F\left[\begin{matrix}2a_t, a_t+a_r \\ 1+a_t-a_r\end{matrix}; \frac{1}{x}\right].$$

If as in 2.2. we set $x = 1$ after differentiation then we have the operational equation

(2) $$[\Pi(\delta_1+a_r) - \Pi(\delta_1-a_r)] F\left[\begin{matrix}2a_1, a_1+a_r \\ 1+a_1-a_r\end{matrix};\right] = 0.$$

The suffix $_1$ which we will usually omit indicates the existence of a multiplying factor x^{a_1} in front of the hypergeometric series. The effect of this factor is to make δ_1 equivalent to $\delta+a_1$ and we may replace equations 2.2.(1) - (4) by the following,

(3) $$(\delta_1-a_1) F = \frac{2a_1 \Pi(a_1+a_r)}{\Pi(a_1-a_r)} F\left[\begin{matrix}2a_1+1, a_1+a_r+1 \\ 2+a_1-a_r\end{matrix};\right],$$

the products being taken over $r = 2, 3, \cdots s.$,

(4) $$(\delta_1+a_t) F = (a_1+a_t) F\left[\begin{matrix}2a_1, a_1+a_t+1 \\ 1+a_1-a_t\end{matrix};\right],$$

where only the numerator parameter containing a_t is increased by unity,

(5) $$(\delta_1-a_t) F = (a_1-a_t) F\left[\begin{matrix}2a_1, a_1+a_t \\ a_1-a_t\end{matrix};\right]$$

where only the denominator parameter containing a_t is decreased by unity, and

(6) $$\delta_1 F = a_1 F\left[\begin{matrix}2a_1, a_1+1, a_1+a_r \\ a_1, 1+a_1-a_r\end{matrix};\right].$$

From these equations follow immediately, with an obvious notation,

$$(7)\quad (\delta_1^2 - a_1^2)\, F(a_1, a_t) = \frac{2a_1(2a_1+1)\prod(a_1+a_r)}{\prod(a_1-a_r)}\, F(a_1+1, a_t), \text{ and}$$

$$(8)\quad (\delta_1^2 - a_t^2)\, F(a_1, a_t) = (a_1^2 - a_t^2)\, F(a_1, a_t+1),$$

the importance of these operations being that the series on the right hand side of (7) and (8) are both still 'well-poised'.

From equations (2), (6), (7) and (8) we may readily deduce the body of results concerning reducible 'well-poised' series. In what follows it is assumed that there are restrictions on the parameters so as to make the analysis valid and that these restrictions are removed from the final result by analytic continuation, where such a process is permissable.

2.7. KUMMER'S THEOREM.

The series ${}_2F_1\left[\begin{matrix}2a_1, a_1+a_2\\ 1+a_1-a_2\end{matrix}; -1\right]$ is annihilated by the operator

$$[(\delta+a_1)(\delta+a_2) + (\delta-a_1)(\delta-a_2)],$$

which may be written

$$2\,[(\delta^2 - a_2^2) + a_2(a_1+a_2)].$$

Using 2.6.(8). we have therefore

$$(a_1^2 - a_2^2)\, {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2+1\\ a_1-a_2\end{matrix}; -1\right] + a_2(a_1+a_2)\, {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2\\ 1+a_1-a_2\end{matrix}; -1\right] = 0,$$

or writing $a_2 - 1$ for a_2

$$(1)\quad {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2\\ 1+a_1-a_2\end{matrix}; -1\right] = \frac{(1-a_2)}{(1+a_1-a_2)}\, {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2-1\\ 1+a_1-a_2+1\end{matrix}; -1\right],$$

$$= \frac{(1-a_2)_n}{(1+a_1-a_2)_n}\, {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2-n\\ 1+a_1-a_2+n\end{matrix}; -1\right],$$

on repeating the process n times.

To complete the proof we may now consider, in the light of Carlson's theorem, the following function of the complex variable z,

$$f(z) = {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2-z\\ 1+a_1-a_2+z\end{matrix};-1\right] - \frac{\Gamma(1-a_2)\,\Gamma(1+a_1-a_2+z)}{\Gamma(1-a_2+z)\,\Gamma(1+a_1-a_2)}\,{}_2F_1\left[\begin{matrix}2a_1, a_1+a_2\\ 1+a_1-a_2\end{matrix};-1\right].$$

We first show that if a_1 and a_2 are real and

(2) $$-1 < 2a_1 < 0,$$

(3) $$2a_2 < 1,$$

then ${}_2F_1\left[\begin{matrix}2a_1, a_1+a_2-z\\ 1+a_1-a_2+z\end{matrix};-1\right]$ is regular and bounded for $R(z) \geqslant 0$.

The series converges if $R(1-a_1-a_2+z) > 0$; it is regular, $R(z) \geqslant 0$, if $(1+a_1-a_2) > 0$ both of which conditions are ensured by (2) and (3).

The series $\sum_{r=0}^{\infty} \frac{(2a_1)_r}{r!} = (1+2a_1) - \sum_{r=2}^{\infty} \frac{(-)(2a_1)_r}{r!}$, and by (2)

$\sum_{r=2}^{\infty} \frac{(-)(2a_1)_r}{r!}$ is a convergent series of positive terms.

$$\text{Let} \quad u(r) = \frac{(a_1+a_2-z)_r(-)^r}{(1+a_1-a_2+z)_r}.$$

$$\text{Then} \quad \left|\frac{u(r+1)}{u(r)}\right| = \left|\frac{z-(a_1+a_2+r)}{z-(a_2-a_1-r-1)}\right|,$$

which is less than unity, $R(z) \geqslant 0$, provided $a_1+a_2+r < 1+a_1-a_2+r$, i.e. provided condition (3) holds.

Therefore $\sum_{r=2}^{\infty} \frac{(-)(2a_1)_r}{r!}|u(r)| < \sum_{r=2}^{\infty} \frac{(-)(2a_1)_r}{r!}$, and from this it follows that $\sum_{r=2}^{\infty} \frac{(2a_1)_r}{r!}u(r)$ and hence that ${}_2F_1\left[\begin{matrix}2a_1, a_1+a_2-z\\ 1+a_1-a_2+z\end{matrix};-1\right]$ is absolutely and uniformly convergent for $R(z) \geqslant 0$, and is therefore bounded.

Also $\frac{\Gamma(1-a_2)\Gamma(1+a_1-a_2+z)}{\Gamma(1-a_2+z)\Gamma(1+a_1-a_2)}$ is regular for large z provided $(1+a_1-a_2) > 0$ and is $O(|z|^{a_1})$.

Thus $f(z)$ satisfies conditions (1) and (2) of Carlson's theorem 2.3., and by equation (1) $f(n)=0$, $n=0,1,2,\cdots$: Therefore condition (iii) of Carlson's theorem is also satisfied, and $f(z)=0$ identically. In particular if we set $z=a_1+a_2$ we obtain Kummers theorem,

$$ {}_2F_1\left[\begin{matrix}2a_1, a_1+a_2\\ 1+a_1-a_2\end{matrix};-1\right] = \frac{\Gamma(1+a_1)\Gamma(1+a_1-a_2)}{\Gamma(1+2a_1)\Gamma(1-a_2)}. $$

So far we have only proved this subject to conditions (2) and (3) with a_1 and a_2 real, but these restrictions may be relaxed, and the result is true provided that the series converges.

2.8. DIXON'S THEOREM AND RELATED THEOREMS.

By considering the operators

$$ \left[\prod_{r=1}^{3}(\delta+a_r)-\prod_{r=1}^{3}(\delta-a_r)\right] \equiv \left[A(\delta^2-a_3^2)+B\right], $$

$$ \left[\prod_{r=1}^{3}(\delta+a_r)+\prod_{r=1}^{3}(\delta-a_r)\right] \equiv \delta\left[A(\delta^2-a_3^2)+B\right], $$

$$ \left[\prod_{r=1}^{4}(\delta+a_r)-\prod_{r=1}^{4}(\delta-a_r)\right] \equiv \delta\left[A(\delta^2-a_4^2)+B\right], $$

we may obtain the two-term recurrence relations satisfied by ${}_3F_2\left[\begin{matrix}2a_1, a_1+a_r\\ 1+a_1-a_r\end{matrix};\right]$, ${}_4F_3\left[\begin{matrix}2a_1, a_1+1, a_1+a_r\\ a_1, 1+a_1-a_r\end{matrix};-1\right]$, ${}_5F_4\left[\begin{matrix}2a_1, a_1+1, a_1+a_r\\ a_1, 1+a_1-a_r\end{matrix};\right]$, respectively, and as in the proof of Kummer's theorem we may deduce Dixon's theorem and the equivalent theorems for the ${}_4F_3$ and the ${}_5F_4$.[1] In each case however a final stage in the argument involves the use of Carlson's theorem or some limiting process,[2] and makes the proof rather long. Since all these results can be

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(1) Bailey's tract pps. 27, 28.

(2) See for example Bailey. 3.

deduced from Dougall's theorem, in the proof of which no such difficulties arise, we omit the details of these proofs, and proceed immediately to Dougall.

2.9. DOUGALL'S THEOREM. When we consider operators of higher order than those of the last section it becomes apparent that some restriction must be placed on the parameters in order that the operator may be rearranged to give a suitable form.

Consider for example

(1) $$\left[\prod_{r=1}^{6} (\delta + a_r) - \prod_{r=1}^{6} (\delta - a_r) \right] F = 0,$$

satisfied by ${}_6F_5\left[\begin{matrix} 2a_1, a_1+a_r \\ 1+a_1-a_r \end{matrix} ; \right]$, $\quad r = 2,3,4,5,6.$

The operator in (1) is of the form

$$2\delta \left[A\delta^4 + B\delta^2 + C \right],$$

where $A = \sum_{r=1}^{6} a_r$. If then we set

(2) $$\Sigma a_r = 0,$$

and since this is inconsistent with the convergence of all the series considered, we suppose that $a_1 + a_6 = -n$, a negative integer, then the operator may be written

(3) $$2\delta \left[P(\delta^2 - a_5^2) + Q(\delta^2 - a_6^2) \right].$$

Setting $\delta = a_6, a_5$ in (1) and (3) we may evaluate $P + Q$,

$$(a_6 - a_5) P = \prod_{r=1}^{4} (a_6 + a_r) \quad ; \quad (a_5 - a_6) Q = \prod_{r=1}^{4} (a_5 + a_r).$$

From 2.6. (6) and (8) we thus obtain

$$(a_6+a_2)(a_6+a_3)(a_6+a_4)(a_1-a_5)\; F(a_5+1, a_6)$$
$$= (a_5+a_2)(a_5+a_3)(a_5+a_4)(a_1-a_6)\; F(a_5, a_6+1)$$

where $F(a_5, a_6) = {}_7F_6\left[\begin{matrix} 2a_1, a_1+1, a_1+a_r \\ a_1, 1+a_1-a_r \end{matrix};\right]$.

Writing $a_5 - 1$ for a_5 we have

$$F(a_5, a_6) = \frac{(1-a_2-a_5)(1-a_3-a_5)(1-a_4-a_5)(a_6-a_1)}{(a_6+a_2)(a_6+a_3)(a_6+a_4)(1+a_1-a_5)} F(a_5-1, a_6+1),$$

$$= \frac{(1-a_2-a_5)_n(1-a_3-a_5)_n(1-a_4-a_5)_n(a_6-a_1)_n}{(a_6+a_2)_n(a_6+a_3)_n(a_6+a_4)_n(1+a_1-a_5)_n} F(a_5-n, a_6+n),$$

and since $F(a_5-n, a_6+n)$ is equal to unity

$$ {}_7F_6\left[\begin{matrix} 2a_1, a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5, & -n \\ a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5, & 1+2a_1+n \end{matrix};\right]$$

$$= \frac{(1-a_2-a_5)_n(1-a_3-a_5)_n(1-a_4-a_5)_n(1+2a_1)_n}{(1+a_1-a_2)_n(1+a_1-a_3)_n(1+a_1-a_4)_n(1+a_1-a_5)_n},$$

$$= \frac{(1-a_3-a_4)_n(1-a_2-a_4)_n(1-a_2-a_3)_n(1+2a_1)_n}{(1+a_1-a_2)_n(1+a_1-a_3)_n(1+a_1-a_4)_n(1-a_1-a_2-a_3-a_4)_n},$$

subject to the condition $a_2+a_3+a_4+a_5 = 1+n$.

This is Dougall's theorem.

2.10 A REDUCIBLE WELL-POISED ${}_7F_6$

We now prove the following result,

if $a_2+a_3+a_4+a_5 = m$, $a_1+a_6 = -m$, then

(1) $${}_7F_6\left[\begin{matrix} 2a_1, a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5, & a_1+a_6 \\ a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5, & 1+a_1-a_6 \end{matrix};\right]$$

$$= \frac{-\sigma_3(1-a_2-a_5)_{m-1}(1-a_3-a_5)_{m-1}(1-a_4-a_5)_{m-1}(1+2a_1)_m}{(1+a_1-a_2)_m(1+a_1-a_3)_m(1+a_1-a_4)_m(1+a_1-a_5)_m},$$

where σ_3 is the third order elementary symmetric function of $a_1, a_2, \cdots a_6$.

In the ${}_7F_6$ the denominator parameters exceed the numerator parameters by four.

To prove this result we consider the same operator as that used in the previous section, namely

(2) $$\left[\prod_{r=1}^{6}(\delta+a_r) - \prod_{r=1}^{6}(\delta-a_r)\right] \equiv 2\delta\left[\sigma_1\delta^4+\sigma_3\delta^2+\sigma_5\right].$$

Since $\sigma_1 = 0$, this operator may be written

$$2\delta\left[\sigma_3(\delta^2-a_5^2) + (a_5+a_1)(a_5+a_2)(a_5+a_3)(a_5+a_4)(a_5+a_6)\right],$$

and therefore

$$(3)\quad (a_5-a_1)\sigma_3\,{}_7F_6\left[\begin{matrix}2a_1, a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & 1+a_1+a_5, & -m\\ a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & a_1-a_5, & 1+2a_1+m\end{matrix};\right]$$

$$= (a_5+a_2)(a_5+a_3)(a_5+a_4)(a_5+a_6)$$

$${}_7F_6\left[\begin{matrix}2a_1, a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5, & -m\\ a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5, & 1+2a_1+m\end{matrix};\right].$$

In the series on the left hand side of this equation the sum of the denominator parameters exceeds the sum of the numerator parameters by two, and this series is therefore reducible by Dougall's theorem, giving the required result.

If in addition to the vanishing of σ_1, σ_3 also vanishes then the series is equal to zero, a fact which is immediately obvious from the form of the operator (2).

Since the result of this section appears to be new, we add a second proof by Dougall's method.

Consider the equation

$$(4)\quad {}_7F_6\left[\begin{matrix}2a_1, a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5, & a_1+a_6\\ a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5, & 1+a_1-a_6\end{matrix};\right]$$

$$= \frac{-\sigma_3\,\Gamma(a_3+a_4)\Gamma(a_2+a_4)\Gamma(a_2+a_3)\Gamma(1+a_1-a_2)\Gamma(1+a_1-a_3)\Gamma(1+a_1-a_4)\Gamma(1+a_1-a_5)\Gamma(1+a_1-a_6)}{\Gamma(1+2a_1)\Gamma(1-a_2-a_5)\Gamma(1-a_3-a_5)\Gamma(1-a_4-a_5)\Gamma(1-a_2-a_6)\Gamma(1-a_3-a_6)\Gamma(1-a_4-a_6)\Gamma(1-a_5-a_6)},$$

together with the condition $\sigma_1 = 0$.

If $a_1+a_6 = 0$ the left hand side of this equation is equal to unity, the right hand side reduces to

$$\frac{-\sigma_3}{(a_3+a_4)(a_2+a_4)(a_2+a_3)}$$

i.e. is equal to $\dfrac{-\{a_1a_6\sigma_1' + (a_1+a_6)\sigma_2' + \sigma_3'\}}{(a_3+a_4)(a_2+a_4)(a_2+a_3)}$,

where σ' refers to a_2, a_3, a_4, a_5, and $\sigma_1' = -(a_1+a_6) = 0$, so that the expression is equal to

$$\frac{-\sigma_3'}{(a_3+a_4)(a_2+a_4)(a_2+a_3)},$$

$$= \frac{(a_2+a_3+a_4)(a_2a_3+a_2a_4+a_3a_4) - a_2a_3a_4}{(a_3+a_4)(a_2+a_4)(a_2+a_3)},$$

$$= 1.$$

The equation is therefore satisfied by $a_6 = -a_1$, and we will suppose that it is satisfied when $a_6 = -a_1, -a_1-1, \cdots, -a_1-m+1$. Then by symmetry it is also satisfied if a_4 or a_3 have any of these m-values, that is for a total of $2m$ values of a_4, namely

$$a_4 = -a_1-r, \ (r = 0, 1, \cdots, m-1) \ ; \ a_4 = -a_6-a_2-a_5+s, \ (s = 0, 1, \cdots m-1).$$

It is therefore true in particular when $a_1+a_6 = -m$, and a_4 has one of $2m$ values. Setting $a_1+a_6 = -m$ equation (4) becomes

$${}_7F_6\left[\begin{matrix} 2a_1, a_1+1, a_1+a_2, a_1+a_3, a_1+a_4, a_1+a_5, -m \\ a_1, 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5, 1+2a_1+m \end{matrix};\right]$$

$$= \frac{-\sigma_3(1-a_2-a_5)_{m-1}(1-a_3-a_5)_{m-1}(1-a_4-a_5)_{m-1}(1+2a_1)_m}{(1+a_1-a_2)_m(1+a_1-a_3)_m(1+a_1-a_4)_m(1+a_1-a_5)_m}.$$

Multiplying both sides of this relation by $(1+a_1-a_3)_m(1+a_1-a_4)_m$, we then have an equation connecting two polynomials of degree $2m$ in a_4, since we are considering $a_3 = -(a_1+a_2+a_4+a_5+a_6)$ and therefore σ_3 is of degree two in a_4. We know that this equation is satisfied for $2m$ values of a_4; if it is true for one other value it is an identity, and the theorem is proved. Setting $a_4 = 1-a_5$ and the right hand side vanishes whilst the left hand side becomes a ${}_5F_4$

which is reducible, and which vanishes under the condition[1]

$$1+a_2+a_3 = m .$$

If in (1) we substitute for a_5 then σ_3 is the third order elementary symmetric function of the quantities $a_1, a_2, a_3, a_4, m-a_2-a_3-a_4, -a_1-m$ and therefore $-\sigma_3 = \{m^2(a_1+a_2+a_3+a_4)+\cdots\}$. Now let $m\to\infty$ and we obtain the known result

$$ {}_5F_4\left[\begin{matrix}2a_1, & a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4\\ & a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4\end{matrix};\right] = \frac{\Gamma(1+a_1-a_2)\Gamma(1+a_1-a_3)\Gamma(1+a_1-a_4)\Gamma(1-a_1-a_2-a_3-a_4)}{\Gamma(1+2a_1)\Gamma(1-a_3-a_4)\Gamma(1-a_2-a_4)\Gamma(1-a_2-a_3)} .$$

2.11. FURTHER EXAMPLES OF REDUCIBLE SERIES.

The result of the last section was obtained by finding a relation between a series which is known to be reducible, in this case the Dougall ${}_7F_6$, and an allied series. Other examples of such pseudo-reducible series may be obtained when the related series is either a ${}_2F_1$ or a Saalschützian ${}_3F_2$.

Thus the operator

$$[(\delta_1+a_1)(\delta_1+b)-(\delta_1-a)(\delta_1-c)] , \qquad \delta_1 \equiv \delta+a$$

which annihilates ${}_2F_1\left[\begin{matrix}2a, a+b\\ 1+a-c\end{matrix};\right]$, may be written

$$[(2a+b+c)\delta_1 + a(b-c)],$$

and hence by 2.6.(6)

(1) $$ {}_3F_2\left[\begin{matrix}2a, & a+1, & a+b\\ & a, & 1+a-c\end{matrix};\right] = \frac{(c-b)}{(2a+b+c)}\,{}_2F_1\left[\begin{matrix}2a, a+b\\ 1+a-c\end{matrix};\right] = \frac{(c-b)}{(2a+b+c)}\,\frac{\Gamma(1+a-c)\Gamma(1-2a-b-c)}{\Gamma(1-a-c)\Gamma(1-b-c)} ,$$

[1] tract 4.4.(1).

which is a special case of 1.4.(6). If in (1) we set $a+b=-m$ we obtain

$$(2)\qquad {}_3F_2\left[\begin{matrix}2a, a+1, -m\\ a, 1+a-c\end{matrix};\right] = -\frac{(a+c+m)(1-a-c)_{m-1}}{(1+a-b)_m}\;^{(1)}.$$

To obtain pseudo-reducible series allied to the Saalschützian ${}_3F_2$ we consider the operator

$$[(\delta_1+a)(\delta_1+b)(\delta_1-a-m)-(\delta_1-a)(\delta_1-b)(\delta_1+a+2b-m)],\qquad \delta_1\equiv\delta+a,$$

are annihilated satisfied by $\quad {}_3F_2\left[\begin{matrix}2a, a+b, -m\\ 1+a-b, 1+2a+2b-m\end{matrix};\right].$

This operator may be written in either of the forms

$$(3)\qquad 2(a+b)[(b-m)(\delta_1-b)-b(a-b+m)],$$

$$(4)\qquad 2(a+b)[(b_1-m)\delta_1-ab].$$

The first of these gives

$$(b-m)(a-b)\,{}_3F_2\left[\begin{matrix}2a, a+b, -m\\ a-b, 1+2a+2b-m\end{matrix};\right] = b(a-b+m)\,{}_3F_2\left[\begin{matrix}2a, a+b, -m\\ 1+a-b, 1+2a+2b-m\end{matrix};\right],$$

where the series on the left is Saalschützian, and so we have

$$(5)\qquad {}_3F_2\left[\begin{matrix}2a, a+b, -m\\ a-b, 1+2a+2b-m\end{matrix};\right] = \frac{(-2b)_m(-a-b)_m(1-b)_m}{(1+a-b)_m(-2a-2b)_m(-b)_m}\;^{(2)}.$$

From (4)

$$(b-m)\,{}_4F_3\left[\begin{matrix}2a, a+1, a+b, -m\\ a, 1+a-b, 1+2a+2b-m\end{matrix};\right] = b\,{}_3F_2\left[\begin{matrix}2a, a+b, -m\\ 1+a-b, 1+2a+2b-m\end{matrix};\right],$$

the right hand side of which may be summed by (5) to give

$$(6)\qquad {}_4F_3\left[\begin{matrix}2a, a+1, a+b, -m\\ a, 1+a-b, 1+2a+2b-m\end{matrix};\right] = \frac{(-2b)_m(-a-b)_m}{(1+a-b)_m(-2a-2b)_m}\;^{(3)}.$$

As a last example of this type of reducible series, consider

$$(7)\qquad [(\delta_1-a)(\delta_1-b)(\delta_1+1+a+2b-m)-(\delta_1+a)(\delta_1+b)(\delta_1-a-m)],$$

(1) (2) (3). Bailey's tract, page 30.

the operator belonging to the series ${}_3F_2\left[\begin{matrix}2a, a+b, -m \\ 1+a-b, 2+2a+2b-m\end{matrix};\right]$.

We write this operator in the form

(8) $\left[(\delta_1-a-m)(\delta_1+1+a+2b-m)+A\,\delta_1+B(\delta_1-a-m)\right]$

to obtain the relation

(9) $$-m(1+2a+2b-m)\,{}_3F_2\left[\begin{matrix}2a, a+b, -m+1 \\ 1+a-b, 1+2a+2b-m\end{matrix};\right]$$

$$+A.a.\,{}_4F_3\left[\begin{matrix}2a, a+1, a+b, -m \\ a, 1+a-b, 2+2a+2b-m\end{matrix};\right]$$

$$-m.B.\,{}_3F_2\left[\begin{matrix}2a, a+b, -m+1 \\ 1+a-b, 2+2a+2b-m\end{matrix};\right]=0.$$

Setting $\delta_1=(a+m)$, $\delta_1=0$ in (7) and (8) we can evaluate A and B,

$$(a+m)A = m(m+a-b)(1+2a+2b),$$

$$-(a+m)\,B = ab(1+2a+2b)+(a+m)(1+a+2b-m),$$

and since in (9) the first ${}_3F_2$ is Saalschützian and the second ${}_3F_2$ is reducible by equation (5) of this section, we obtain after a little reduction

$${}_4F_3\left[\begin{matrix}2a, a+1, a+b, -m \\ a, 1+a-b, 2+2a+2b-m\end{matrix};\right]=\frac{(-1-a-b)_m(-1-2b)_m(-b+\frac{1}{2})_m}{(1+a-b)_m(-1-2a-2b)_m(-b-\frac{1}{2})_m}.\ ^{(1)}$$

2.12. REDUCIBLE BILATERAL SERIES. The bilateral series

$$H\left[\begin{matrix}a_r \\ b_s\end{matrix};x\right]=\sum_{-\infty}^{+\infty}\frac{\prod(a_r)_n}{\prod(b_s)_n}x^n, \qquad |x|=1,$$

where $(a_r)_{-n}=\dfrac{(-)^n}{(1-a_r)_n}$, satisfies the following operational relations,

$$(\delta+a_r)H\left[\begin{matrix}a_r \\ b_s\end{matrix};x\right]=a_r H\left[\begin{matrix}a_r+1 \\ b_s\end{matrix};x\right],$$

$$(\delta+b_s-1)H\left[\begin{matrix}a_r \\ b_s\end{matrix};x\right]=(b_s-1)H\left[\begin{matrix}a_r \\ b_s-1\end{matrix};x\right],$$

where only the parameters specialised are augmented or diminished,

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(1) Bailey's tract. p. 30. 4.5.(1.4).

and $$x\,H\left[\begin{matrix}a_r\\ b_s\end{matrix};x\right] = \frac{\Pi(b_s-1)}{\Pi(a_r-1)}\,H\left[\begin{matrix}a_r-1\\ b_s-1\end{matrix};x\right],$$

where every parameter is diminished by unity in the series on the right hand side of this relation. From these relations we may deduce that $H \equiv H\left[\begin{matrix}a_r\\ b_s\end{matrix};x\right]$ satisfies

(1) $$\left[\Pi(\delta+b_s-1) - x\,\Pi(\delta+a_r)\right]H = 0.$$

If we set $x = 1$ after the differentiations have been performed then we have as an operator which annihilates $H\left[\begin{matrix}a_r\\ b_s\end{matrix};\right]$,

(2) $$\left[\Pi(\delta+b_s-1) - \Pi(\delta+a_r)\right],$$

and we may investigate reducible bilateral series using the same technique that we have used throughout this chapter.

Thus immediately
$$_1H_1\left[\begin{matrix}a\\ b\end{matrix};\right] = 0,$$

since the series satisfies
$$\left[(\delta+b-1)-(\delta+a)\right]H = 0,$$
i.e. $$(b-a-1)\,H = 0.$$

To obtain the bilateral analogue of Gauss' theorem we consider the operator $\left[(\delta+c)(\delta+d)-(\delta+a)(\delta+b)\right]$ belonging to the series $_2H_2\left[\begin{matrix}a,\ b\\ c+1,\ d+1\end{matrix};\right]$. This operator may be written in either of the forms.

$$\left[(c+d-a-b)(\delta+c) - (a-c)(b-c)\right],$$
$$\left[(c+d-a-b)(\delta+a) + (c-a)(d-a)\right],$$

from which we may deduce the two relations,

$${}_2H_2\left[\begin{matrix}a, b\\ c, d+1\end{matrix};\right] = \frac{(a-c)(b-c)}{c(c+d-a-b)}{}_2H_2\left[\begin{matrix}a, b\\ c+1, d+1\end{matrix};\right],$$

$${}_2H_2\left[\begin{matrix}a+1, b\\ c+1, d+1\end{matrix};\right] = \frac{(a-c)(d-a)}{a(c+d-a-b)}{}_2H_2\left[\begin{matrix}a, b\\ c+1, d+1\end{matrix};\right].$$

We may combine these results to obtain

$${}_2H_2\left[\begin{matrix}a, b\\ c, d\end{matrix};\right] = \frac{(c-a)(c-a+1)(d-a)(c-b)}{c(1-a)(c+d-a-b-1)(c+d-a-b)}{}_2H_2\left[\begin{matrix}a-1, b\\ c+1, d\end{matrix}\right],$$

and repeating the process

(3) $${}_2H_2\left[\begin{matrix}a, b\\ c, d\end{matrix};\right] = \frac{(c-a)_{2n}(d-a)_n(c-b)_n}{(1-a)_n(c)_n(c+d-a-b-1)_{2n}}{}_2H_2\left[\begin{matrix}a-n, b\\ c+n, d\end{matrix};\right].$$

Consider now the function

(4) $$f(z) = {}_2H_2\left[\begin{matrix}a, b\\ c, d\end{matrix};\right] - \frac{\Gamma(c-a+2z)\Gamma(d-a+z)\Gamma(c-b+z)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)}\cdot \frac{\Gamma(1-a)\Gamma(c)\Gamma(c+d-a-b-1)}{\Gamma(1-a+z)\Gamma(c+z)\Gamma(c+d-a-b-1+2z)}{}_2H_2\left[\begin{matrix}a-z, b\\ c+z, d\end{matrix};\right].$$

By a line of argument parallel to that used in the proof of Kummer's theorem 2.7. it can be shown that all the conditions of Carlson's theorem are satisfied by $f(z)$, provided that the parameters satisfy the relations

(i) $R(d-b-1) > 0$, (ii) a and c real, $c > a$, $a < 1$.

$f(z)$ is therefore identically equal to zero, and in particular setting $z = 1-c$ ($c-1$), when ${}_2H_2\left[\begin{matrix}a-z, b\\ c+z, d\end{matrix};\right]$ becomes ${}_2F_1\left[\begin{matrix}1+a-c, b\\ d\end{matrix};\right]$, we obtain the result

(5) $${}_2H_2\left[\begin{matrix}a, b\\ c, d\end{matrix};\right] = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(c)\Gamma(d)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}\,[1].$$

This result may be obtained in another way. The ${}_2H_2\left[\begin{matrix}a, b\\ c, d\end{matrix};x\right]$ is the sum of two solutions ${}_3F_2\left[\begin{matrix}1, a, b\\ c, d\end{matrix};x\right]$ and

[1] Bailey. 2. p. 105. (1.3).

$\frac{(1-c)(1-d)}{(1-a)(1-b)}\frac{1}{x}\,{}_3F_2\left[\begin{matrix}1, 2-c, 2-d\\ 2-a, 2-b\end{matrix};\frac{1}{x}\right]$ of the differential equation

$$\delta\left[(\delta+c-1)(\delta+d-1)-x(\delta+a)(\delta+b)\right]y=0.$$

Other solutions of this equation in descending powers of x are solutions of the inner bracket equated to zero $[(\delta+c-1)(\delta+d-1)-x(\delta+a)(\delta+b)]y=0$, namely $x^{-a}\,{}_2F_1\left[\begin{matrix}1+a-c, 1+a-d\\ 1+a-b\end{matrix};\frac{1}{x}\right]$ and $x^{-b}\,{}_2F_1\left[\begin{matrix}1+b-c, 1+b-d\\ 1+b-a\end{matrix};\frac{1}{x}\right]$.

On evaluating the integral $\frac{1}{2\pi i}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\int\frac{\Gamma(1+s)\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)\Gamma(d+s)}(-x)^s ds$ in the usual way round an infinite circle we obtain a relation between these four solutions

$$ {}_3F_2\left[\begin{matrix}1, a, b\\ c, d\end{matrix};x\right]+\frac{(1-c)(1-d)}{(1-a)(1-b)}x^{-1}\,{}_3F_2\left[\begin{matrix}1, 2-c, 2-d\\ 2-a, 2-b\end{matrix};\frac{1}{x}\right]$$

$$=\frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)\Gamma(d-a)}e^{i\pi a}x^{-a}\,{}_2F_1\left[\begin{matrix}1+a-c, 1+a-d\\ 1+a-b\end{matrix};\frac{1}{x}\right]$$

+ a similar expression with a and b interchanged.

Setting $x=1$ in this relation, the ${}_2F_1$'s on the right are reducible by Gauss' theorem, and (5) follows easily.[1]

As an example of the technique applied to a reducible 'well-poised' bilateral series, we prove the result

(6) $${}_5H_5\left[\begin{matrix}a_1+1, a_1+a_2, a_1+a_3, a_1+a_4, a_1+a_5\\ a_1, 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5\end{matrix};\right]$$

$$=\frac{\Gamma(1-a_1-a_2)\Gamma(1-a_1-a_3)\Gamma(1-a_1-a_4)\Gamma(1-a_2-a_3-a_4-a_5)}{\Gamma(1-a_2-a_5)\Gamma(1-a_3-a_5)\Gamma(1-a_4-a_5)\Gamma(1-2a_1)}\cdot$$

$$\cdot\frac{\Gamma(1+a_1-a_5)\Gamma(1-a_1-a_5)\Gamma(1+a_1-a_2)\Gamma(1+a_1-a_3)\Gamma(1+a_1-a_4)}{\Gamma(1+2a_1)\Gamma(1-a_2-a_3)\Gamma(1-a_2-a_4)\Gamma(1-a_3-a_4)}.$$ [2]

${}_4H_4\left[\begin{matrix}a_1+a_2, a_1+a_3, a_1+a_4, a_1+a_5\\ 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5\end{matrix};\right]$ is annihilated by the operator $[(\delta_1+a_2)(\delta_1+a_3)(\delta_1+a_4)(\delta_1+a_5)-(\delta_1-a_2)(\delta_1-a_3)(\delta_1-a_4)(\delta_1-a_5)]$, $\delta_1\equiv\delta+a_1$,

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[1] c.f. JACKSON. M. 2.

[2] BAILEY. 2. p. 106. (2.3).

which may be written

$$2\delta_1\left[(a_2+a_3+a_4+a_5)(\delta_1^2-a_5^2)+(a_2+a_5)(a_3+a_5)(a_4+a_5)\right].$$

If $H(a_5) \equiv {}_5H_5\left[\begin{matrix} a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5 \\ a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5 \end{matrix};\right]$ we thus have

$$H(a_5+1) = \frac{-(a_2+a_5)(a_3+a_5)(a_4+a_5)}{(a_1-a_5)(a_1+a_5)(a_2+a_3+a_4+a_5)}\,H(a_5).$$

Thus

$$(7)\qquad H(a_5) = \frac{(1-a_2-a_5)_n(1-a_3-a_5)_n(1-a_4-a_5)_n}{(1-a_2-a_3-a_4-a_5)_n(1+a_1-a_5)_n(1-a_1-a_5)_n}\,H(a_5-n).$$

We now consider the function $f(z)$ equal to

$$H(a_5) - \frac{\Gamma(1-a_2-a_5+z)\Gamma(1-a_3-a_5+z)\Gamma(1-a_4-a_5+z)\Gamma(1-a_2-a_3-a_4-a_5)}{\Gamma(1-a_2-a_5)\Gamma(1-a_3-a_5)\Gamma(1-a_4-a_5)\Gamma(1-a_2-a_3-a_4-a_5+z)}\cdot \frac{\Gamma(1+a_1-a_5)\Gamma(1-a_1-a_5)}{\Gamma(1+a_1-a_5+z)\Gamma(1-a_1-a_5+z)}\,H(a_5-z).$$

This vanishes by (7) when $z=n$, and reduces to the required result (6) when $z=a_5-a_1$. It can be shown that the conditions of Carlson's theorem are satisfied provided that the parameters satisfy the relations (i) $R[1-2(a_2+a_3+a_4)]>0$, (ii) a_1 and a_5 real, $2a_5<1$, $2a_1<1$.

2.13. SERIES WHICH ARE JUST NOT REDUCIBLE. So far we have been concerned with reducible series satisfying two term recurrence relations. If a series is not reducible it will be found to satisfy a many-term recurrence relation when regarded as a function of one of its parameters. Thus for example $F(a_3)\equiv {}_3F_2\left[\begin{matrix} a_1, & a_2, & a_3 \\ & 1+b_1, & 1+b_2\end{matrix};\right]$ satisfies the operational equation

$$[\delta(\delta+b_1)(\delta+b_2)-(\delta+a_1)(\delta+a_2)(\delta+a_3)]\,F(a_3)=0,$$

which may be written

$$[A(\delta+a_3)(\delta+a_3+1)+B(\delta+a_3)+C]\,F(a_3)=0,$$

so that $A(a_3+1)a_3F(a_3+2)+Ba_3F(a_3+1)+CF(a_3)=0$.

Chaundy[1] has considered the recurrence relations satisfied by terminating series, and has obtained such relations when the 'parameter of closure' appears not only as a numerator parameter $-n$, but also in certain of the other parameters in the forms $a\pm n$. In what follows we borrow extensively from Chaundy's techniques and results.

Investigation shows that in known transformations connecting two series such as Whipple's relation or Bailey's relation 1.4. (8), (9), the series concerned invariably satisfy three-term recurrence relations and we may obtain inductive proofs of these identities by showing that the two series involved in the identity satisfy the same relation. We note that the 'δ-method' loses much of its power when applied to three-term difference relations and the proofs which follow are considerably less elegant than those which occured earlier in the chapter.

2.14. VARYING THE COEFFICIENTS IN A RECURRENCE RELATION

If F_n satisfies the recurrence relation

$$f_0(n)\,F_n+f_1(n-1)\,F_{n-1}+f_2(n-2)\,F_{n-2}=0,$$

then $G_n=\frac{(a)_n}{(c)_n}F_n$ satisfies the relation

$$(c+n-1)(c+n-2)f_0(n)\,G_n+(a+n-1)(c+n-2)f_1(n-1)G_{n-1}+(a+n-1)(a+n-2)f_2(n-2)G_{n-2}=0.$$

In particular if $(c+n-2)$ is a factor of $f_2(n-2)$, and $(a+n-1)$ is a factor of $f_0(n)$, then the effect is to exchange end terms, i.e. $(c+n-1)$

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[1] Chaundy. 2.

for $(a+n-1)$ in the coefficient of G_n, and $(a+n-2)$ for $(c+n-2)$ in the coefficient of G_{n-2}, leaving the middle term unaltered. Thus if we can identify the middle coefficients in two three-term recurrence relations then we may be able, by means of this device, to complete the identification.

2.15. A TRANSFORMATION CONNECTING TWO SAALSCHÜTZIAN ${}_4F_3$.

As a first illustration of the method we prove the identity quoted in Bailey's tract[1] as

$$(1) \quad {}_4F_3\left[\begin{matrix}x, y, z, -n \\ u, v, w\end{matrix};\right] = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\,{}_4F_3\left[\begin{matrix}u-x, u-y, z, -n \\ 1-v+z-n, 1-w+z-n, u\end{matrix};\right].$$

Taking advantage of a symmetry latent in this relation we notice that what we have to prove may be stated in the following way,

if (i) $\alpha_1+\alpha_2 = b+1$, (ii) $\beta_1+\beta_2 = a+1$, then

$$(2) \quad \phi(n,\lambda) = (\beta_1+\lambda)_n(\beta_2+\lambda)_n \cdot {}_4F_3\left[\begin{matrix}\alpha_1+\lambda, \alpha_2+\lambda, a+n, -n \\ \beta_1+\lambda, \beta_2+\lambda, b+1\end{matrix};\right]$$

is an even function of λ.

This is certainly true for $n=0$, $n=1$, since $\phi(0,\lambda)=1$, and $(b+1)\,\phi(1,\lambda) = (b+1)(\beta_1+\lambda)(\beta_2+\lambda) - (a+1)(\alpha_1+\lambda)(\alpha_2+\lambda)$ is a quadratic in λ in which the coefficient of λ vanishes due to conditions (i) and (ii).

We now obtain the three-term difference relation satisfied by $\phi(n,\lambda)$ or rather by $\Phi_n = \frac{(a)_n}{n!}\phi(n,\lambda)$, and show that the coefficients in this relation are even functions of λ, to complete the proof by induction.

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[1] tract. 7.2.(1).

The series ${}_4F_3\left[\begin{matrix}\alpha_1+\lambda, \alpha_2+\lambda, a+n-2, -n\\ \beta_1+\lambda, \beta_2+\lambda, b+1\end{matrix};\right]$ is annihilated by the operator

(3) $$\left[\delta(\delta+b)(\delta+\beta_1+\lambda-1)(\delta+\beta_2+\lambda-2)-(\delta+\alpha_1+\lambda)(\delta+\alpha_2+\lambda)(\delta+a+n-2)(\delta-n)\right],$$

which by virtue of conditions (i) and (ii) may be written

(4) $$\left[A(\delta+a+n-2)(\delta+a+n-1) + B(\delta+a+n-2)(\delta-n) + C(\delta-n)(\delta-n+1)\right].$$

Therefore $F_n \equiv {}_4F_3\left[\begin{matrix}\alpha_1+\lambda, \alpha_2+\lambda, a+n, -n\\ \beta_1+\lambda, \beta_2+\lambda, b+1\end{matrix};\right]$ satisfies

(5) $$A(a+n-2)(a+n-1)F_n + B(a+n-2)(-n)F_{n-1} + C(-n)(-n+1)F_{n-2} = 0.$$

A, B, and C may be evaluated by setting $\delta = n, 2-a-n, n-1$, in turn in (3) and (4) and setting $\Phi_n = \frac{(a)_n}{n!}(\beta_1+\lambda)_n(\beta_2+\lambda)_n . F_n$ in (5) and using the device of 2.14. we have eventually

(6) $$\frac{n(n+b)}{(2n+a-2)(2n+a-1)}\Phi_n - B\,\Phi_{n-1}$$
$$+\frac{(n+a-2)(n+a-b-2)}{(2n+a-3)(2n+a-2)}(n+a-\beta_1-\lambda-1)(n+a-\beta_2-\lambda-1)(\beta_1+\lambda+n-2)(\beta_2+\lambda+n-2)\Phi_{n-2} = 0,$$

where $$B = \frac{n(n+b)(n+\beta_1+\lambda-1)(n+\beta_2+\lambda-1)}{(2n+a-1)} - (n+\alpha_1+\lambda-1)(n+\alpha_2+\lambda-1)$$
$$- \frac{(n-1)(n+b-1)(n+\beta_1+\lambda-2)(n+\beta_2+\lambda-2)}{(2n+a-3)}.$$

In (6) the coefficient of Φ_n is independent of λ, and since by (ii)
$$(n+a-\beta_1+\lambda-1) = (n+\lambda+\beta_2-2)$$
the coefficient of Φ_{n-2} is even in λ.

B is a quadratic in λ, in which the coefficient of λ is

$$\frac{n(n+b)}{(2n+a-1)}(2n-2+\beta_1+\beta_2) - \frac{(n-1)(n+b-1)}{(2n+a-3)}(2n-4+\beta_1+\beta_2) - (2n+\alpha_1+\alpha_2-2)$$
$$= n(n+b)-(n-1)(n+b-1)-(2n+b-1) \quad \text{, by (i) and (ii)}$$
$$= 0.$$

Thus B is an even function of λ, and the identity (2) is proved.

2.16. BAILEY'S IDENTITY. In a similar manner we may prove Bailey's identity connecting two well-poised ${}_9F_8$, which may be written in the following form.

Let $$\Phi_m = \frac{(1+\alpha_1-\beta_2)_m(1+\alpha_1-\beta_3)_m(1+\alpha_1-\beta_4)_m}{m!\,(1+2\alpha_1)_m}\,{}_9F_8\left[\begin{matrix}2\alpha_1, \alpha_1+1, \alpha_1+\alpha_s, \alpha_1+\beta_r \\ \alpha_1, 1+\alpha_1-\alpha_s, 1+\alpha_1-\beta_r\end{matrix};\right]$$

$s = 2,3,4; \quad r = 1,2,3,4.$

Then $\Phi_m(\alpha_r-\lambda, \beta_r+\lambda)$ is an even function of λ provided

$$\text{(i)} \sum_{s=1}^{4}\alpha_s = \sum_{r=1}^{4}\beta_r = 1, \quad \text{(ii)} \quad \alpha_1+\beta_1 = -m.^{(1)}$$

We notice first that $\Phi_0 = 1$, and

$$(1+\alpha_1-\alpha_2)(1+\alpha_1-\alpha_3)(1+\alpha_1-\alpha_4)\,\Phi_1(\alpha+\lambda, \alpha-\lambda)$$
$$= \frac{1}{(1+2\alpha_1+2\lambda)}\left[\begin{matrix}(1+\alpha_1-\alpha_2)(1+\alpha_1-\alpha_3)(1+\alpha_1-\alpha_4)(1+\alpha_1-\beta_2+2\lambda)(1+\alpha_1-\beta_3+2\lambda)(1+\alpha_1-\beta_4+2\lambda) \\ -(\alpha_1+\alpha_2+2\lambda)(\alpha_1+\alpha_3+2\lambda)(\alpha_1+\alpha_4+2\lambda)(\alpha_1+\beta_2)(\alpha_1+\beta_3)(\alpha_1+\beta_4)\end{matrix}\right]$$

$(1+2\alpha_1+2\lambda)$ is readily seen to be a factor of the square bracket on the right hand side of this relation, and this bracket being cubic in λ may therefore be written

$$(1+2\alpha_1+2\lambda)\left[4A\lambda^2 + 2B\lambda + C\right].$$

Equating coefficients of $8\lambda^3$ and $4\lambda^2$ we have

$$A = (1+\alpha_1-\alpha_2)(1+\alpha_1-\alpha_3)(1+\alpha_1-\alpha_4) - (\alpha_1+\beta_2)(\alpha_1+\beta_3)(\alpha_1+\beta_4)$$

$$A(1+2\alpha_1) + B = (1+\alpha_1-\alpha_2)(1+\alpha_1-\alpha_3)(1+\alpha_1-\alpha_4)(3+3\alpha_1-\beta_2-\beta_3-\beta_4)$$
$$- (\alpha_1+\beta_2)(\alpha_1+\beta_3)(\alpha_1+\beta_4)(3\alpha_1+\alpha_2+\alpha_3+\alpha_4).$$

By (i) and (ii) $(3+3\alpha_1-\beta_2-\beta_3-\beta_4) = (1+2\alpha_1) = (3\alpha_1+\alpha_2+\alpha_3+\alpha_4)$, and therefore $B = 0$, and Φ_1 is even in λ

(1) This symmetrical form of the identity was noticed by Burchnall.

Exactly as in the previous section we now find the difference relation satisfied by Φ_m, considering the related series

$$\delta_1 G = \frac{(1+\alpha_1-\beta_2)_{m-2}(1+\alpha_1-\beta_3)_{m-2}(1+\alpha_1-\beta_4)_{m-2}}{m!\,(1+2\alpha_1)_m}\; {}_9F_8\left[\begin{matrix}\alpha_1+\alpha_r, & \alpha_1+\beta_r, & \alpha_1+1 & ; \\ 1+\alpha_1-\alpha_r, & 1+\alpha_1-\beta_r, & \alpha_1 & \end{matrix}\right]$$

where now (iii) $\Sigma_1\, \alpha_r = 1$, (iv) $\Sigma_1\, \beta_r = -1$, so that G is annihilated by the operator

(2) $$\left[\Pi(\delta_1+\alpha_r)(\delta_1+\beta_r) - \Pi(\delta_1-\alpha_r)(\delta_1-\beta_r)\right], \qquad \delta_1 \equiv \delta+\alpha_1 .$$

By virtue of conditions (iii) and (iv) this operator is of order $\delta_1[\delta_1^4, \delta_1^2, 1]$ and may thus be written

(3) $$2\delta_1\left[P(\delta_1^2-\beta_1^2)(\delta_1^2-\overline{\beta_1+1}^2) + Q(\delta_1^2-\beta_1^2)(\delta_1^2-\beta_2^2) + R(\delta_1^2-\beta_2^2)(\delta_1^2-\overline{\beta_2+1}^2)\right],$$

where

(4) $$\begin{cases} (\delta_1^2-\beta_1^2)(\delta_1^2-\overline{\beta_1+1}^2)\,\delta_1 G = \Phi_{m-2}(\beta_2), \\[2ex] (\delta_1^2-\beta_1^2)(\delta_1^2-\beta_2^2)\,\delta_1 G = \dfrac{(-)\,(\alpha_1+\beta_2)}{(1+\beta_1+\beta_3)(1+\beta_1+\beta_4)}\,\Phi_{m-1}(\beta_2+1), \\[2ex] (\delta_1^2-\beta_2^2)(\delta_1^2-\overline{\beta_2+1}^2)\,\delta_1 G = \dfrac{(\alpha_1+\beta_2)(\alpha_1+\beta_2+1)}{(\beta_1+\beta_3)(1+\beta_1+\beta_3)(\beta_1+\beta_4)(1+\beta_1+\beta_4)}\,\Phi_m(\beta_2+2). \end{cases}$$

P and R are evaluated by setting $\delta_1 = \beta_2, \beta_1$ in (2) and (3),

(5) $$\begin{cases} P = \dfrac{\Pi(\beta_2+\alpha_r)(\beta_2+\beta_3)(\beta_2+\beta_4)}{(\beta_2-\beta_1)(\beta_2-\beta_1-1)(\beta_2+\beta_1+1)}, \\[2ex] R = \dfrac{\Pi(\beta_1+\alpha_r)(\beta_1+\beta_3)(\beta_1+\beta_4)}{(\beta_1-\beta_2)(\beta_1-\beta_2-1)(\beta_1+\beta_2+1)}. \end{cases}$$

Quantities such as $(\alpha_r+\beta_s)$, $(\alpha_r-\alpha_s)$, $(\beta_r-\beta_s)$, are all invariant if the α are increased by λ, and the β are decreased by λ, and writing products of such factors as I_1, I_2, etc we have easily from (iii), (iv) and (v)

(6) $$\Phi_m - I_1\,Q\,(1+\beta_1+\beta_2)\,\Phi_{m-1} + I_2(1+\beta_1+\beta_3)(1+\beta_1+\beta_4)(\beta_2+\beta_3)(\beta_2+\beta_4)\,\Phi_{m-2} = 0.$$

Also since $\Sigma_1\,\beta_r = -1$,

$(1+\beta_1+\beta_3+2\lambda)(\beta_2+\beta_4+2\lambda) = (\beta_2+\beta_4-2\lambda)(1+\beta_1+\beta_3-2\lambda)$ etc, so that the coefficient of Φ_{m-2} is an even function of λ, and the identity is established if we show that $(1+\beta_1+\beta_2)\,Q$ is also an even function of λ.

In (2) and (3) set $\delta = \beta_3$ and substitute for P and R from (8)

$$\prod(\beta_3+\alpha_r)(\beta_4+\beta_3) = \frac{(\beta_3-\beta_1)(\beta_3-\beta_1-1)(\beta_3+\beta_1+1)\prod(\beta_2+\alpha_r)(\beta_2+\beta_4)}{(\beta_2-\beta_1)(\beta_2-\beta_1-1)(1+\beta_1+\beta_2)}$$

$$+(\beta_3-\beta_1)(\beta_3-\beta_2)\,Q + \frac{(\beta_3-\beta_2)(\beta_3-\beta_2-1)(1+\beta_2+\beta_3)\prod(\beta_1+\alpha_r)(\beta_1+\beta_4)}{(\beta_1-\beta_4)(\beta_1-\beta_2-1)(1+\beta_1+\beta_2)}.$$

Multiplying by $(1+\beta_1+\beta_2)$ we see that the coefficient of Q in this relation, and also the terms not involving Q are invariant for the changes $\beta\pm\lambda$; therefore Q is even in λ and the proof is complete.

2.17. WHIPPLE'S RELATION. Whipple's relation may of course be deduced from Bailey's.[1] An induction proof on the lines above can be obtained by considering the operator

$$\Big[\prod_{r=1}^{6}(\delta+a_r)-\prod_{r=1}^{6}(\delta-a_r)\Big]$$
$$\equiv 2\delta\big[P(\delta^2-a_6^2)(\delta^2-\overline{a_6+1}^2)+Q(\delta^2-a_6^2)+R\big],$$

leading to

(1) $(a_1^2-a_6^2)(a_1^2-\overline{a_6+1}^2)\,P\,F(a_6+2) + (a_1^2-a_6^2)\,Q\,F(a_6+1) + R\,F(a_6) = 0$

where $F(a_6) = {}_7F_6\left[\begin{matrix}2a_1, & a_1+1, & a_1+a_2, & a_1+a_3, & a_1+a_4, & a_1+a_5, & a_1+a_6\\ & a_1, & 1+a_1-a_2, & 1+a_1-a_3, & 1+a_1-a_4, & 1+a_1-a_5, & 1+a_1-a_6\end{matrix};\right]$.

(2) Also $\begin{cases} P = \sigma_1^6(a_r), \qquad R = \prod_{r=1}^{5}(a_6+a_r), \\ Q = (a_6^2+\overline{a_6+1}^2)\sigma_1^6 + \sigma_3^6, \end{cases}$

where $\sigma_r^6(a_r)$ is the r^{th} elementary symmetric function of $a_1, a_2, a_3, a_4, a_5, a_6$.

Chaundy has shown[2] that under the Saalschützian condition

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[1] Tract. 4.4.

[2] Chaundy. 2. p. 68-69

$$B_1+B_2+B_3 = A_1+A_2+A_3,$$

$$\phi_n = \frac{(2-B_3)_n}{n!}\,{}_4F_3\left[\begin{matrix}A_1, A_2, A_3, -n\\ 1+B_1, 1+B_2, B_3-n-1\end{matrix};\right]$$

satisfies the relation

(3) $$f(n)\,\phi_n - \Delta\,\phi_{n-1} + g(n-B_3)\,\phi_{n-2} = 0,$$

(4) where $$\begin{cases} f(n) = n(n+B_1)(n+B_2), \quad g(n) = (n+A_1)(n+A_2)(n+A_3) \\ \Delta = g(n-1) + (B_3-1)f(n-1) - (B_3-2)f(n) \end{cases}$$

In order to identify the recurrence relations (1) and (3) we first of all obtain Q in a form similar to Δ.

Let σ_r^4 be the r^{th} elementary symmetric function of a_1, a_2, a_3, a_6, and σ_r^3 be the r^{th} elementary symmetric function of a_1, a_2, a_3, and note that $\sigma_3^6 = \sigma_3^4 + (a_4+a_5)\sigma_2^4 + a_4a_5\sigma_1^4$, etc.

Then $$Q = (a_6^2+\overline{a_6+1}^2)\sigma_1^6 + \sigma_3^6$$

$$= \sigma_1^4(a_6^2+\overline{a_6+1}^2) + \sigma_3^4 + (a_4+a_5)(a_6^2+\overline{a_6+1}^2+\sigma_2^4) + a_4a_5\sigma_1^4,$$

$$= \sigma_1^4a_6^2 + \sigma_3^4 + \sigma_1^4(a_6+1-a_4)(a_6+1-a_5) + (a_4+a_5)(a_6^2+\overline{a_6+1}^2+\overline{a_6+1}\,\sigma_1^4+\sigma_2^4),$$

$$= \prod_{r=1}^{3}(a_6+a_r) + \sigma_1^4(a_6+1-a_4)(a_6+1-a_5) + (a_4+a_5)\{a_6^2+a_6(a_6+1)+(a_6+1)^2+(2a_6+1)\sigma_1^3+\sigma_2^3\},$$

$$= \prod_{r=1}^{3}(a_6+a_r) + \sigma_1^4(a_6+1-a_4)(a_6+1-a_5) + (a_4+a_5)\Big\{\prod_{r=1}^{3}(a_6+a_r+1) - \prod_{r=1}^{3}(a_6+a_r)\Big\},$$

so that

(5) $$Q = \sigma_1^4(a_6+1-a_4)(a_6+1-a_5) + (a_4+a_5)\prod_{r=1}^{3}(a_6+a_r+1) - (a_4+a_5-1)\prod_{r=1}^{3}(a_6+a_r),$$

which is the required form.

To obtain Whipples relation in the case where the ${}_7F_6$ terminates we set $a_1+a_6=-n$, and complete the identification of (4) and (5) by the scheme

(6) $$\begin{cases} A_1 = a_1+a_4, \quad A_2 = a_1+a_5, \quad A_3 = 1-a_2-a_3 \\ B_1 = a_1-a_2, \quad B_2 = a_1-a_3, \quad B_3 = 1+a_4+a_5 \end{cases}$$

If we set $\Psi_n = \dfrac{(1+a_1-a_4)_n(1+a_1-a_5)_n}{n!\,(1+2a_1)_n}\,{}_7F_6(-n)$ to make the necessary exchanges of and coefficients, then we obtain from equation (1)

$$n(n+a_1-a_2)(n+a_1-a_3)\Psi_n - Q\Psi_{n-1} + (n+a_1-a_5-1)(n+a_1-a_4-1)(n-a_2-a_3-a_4-a_5)\Psi_{n-2}=0$$

which is identical with Chaundy's equation (3), under the substitutions (6).

Thus ${}_7F_6\left[\begin{matrix}2a_1, a_1+1, a_1+a_2, a_1+a_3, a_1+a_4, a_1+a_5, -n \\ a_1, 1+a_1-a_2, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5, 1+2a_1+n\end{matrix};\right]$, and

$\dfrac{(1+2a_1)_n(1-a_4-a_5)_n}{(1+a_1-a_4)_n(1+a_1-a_5)_n}\,{}_4F_3\left[\begin{matrix}a_1+a_4, a_1+a_5, 1-a_2-a_3, -n \\ 1+a_1-a_2, 1+a_1-a_3, a_4+a_5-n\end{matrix};\right]$ satisfy the same recurrence formula when regarded as functions of n, and comparison when $n=0$, $n=1$ shows that the expressions are identical.

Equation (5) for Q is symmetrical in a_1, a_2, a_3 so that we may set $a_2+a_6=-n$ and identify (4) and (5) by merely interchanging a_1 and a_2 in (6). Although by following the procedure above, this leads to the result that

${}_7F_6\left[\begin{matrix}2a_1, a_1+1, a_1+a_2, a_1-a_2-\cancel{a}, a_1+a_3, a_1+a_4, a_1+a_5 \\ a_1, 1+a_1-a_2, 1+a_1+a_2+n, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5\end{matrix};\right]$ and

$\dfrac{(1+a_2-a_1)_n(1-a_4-a_5)_n}{(1+a_2-a_5)_n(1+a_2-a_4)_n}\cdot{}_4F_3\left[\begin{matrix}a_2+a_4, a_2+a_5, 1-a_1-a_3, -n \\ 1+a_2-a_1, 1+a_2-a_3, a_4+a_5-n\end{matrix};\right]$ both satisfy the same three term difference relation, we are not able to complete the identification, and we obtain Whipple's relation in the case where the ${}_7F_6$ does not terminate by setting $a_6+a_2=n+1$, identifying (4) and (5) by the scheme

$$(7)\quad\left\{\begin{array}{lll}\beta_1 = a_3-a_2, & \beta_2 = a_1-a_2, & \beta_3 = 2-a_4-a_5, \\ A_1 = a_1+a_3, & A_2 = 1-a_2-a_4, & A_3 = 1-a_2-a_5.\end{array}\right.$$

If then we set

$$\Psi_n = \frac{(a_1+a_3+a_4+a_5)_n(1-a_1-a_2)_n}{n!\,(1+a_3-a_2)_n}\,{}_7F_6\left[\begin{matrix}2a_1, a_1+1, a_1+a_2, 1+a_1-a_2+n, a_1+a_3, a_1+a_4, a_1+a_5 \\ a_1, 1+a_1-a_2, a_1+a_2-n, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5\end{matrix};\right]$$

then Ψ_n satisfies the recurrence relation (3) under the substitutions (7), i.e.

$${}_7F_6\left[\begin{matrix}2a_1, a_1+1, a_1+a_2, 1+a_1-a_2+n, a_1+a_3, a_1+a_4, a_1+a_5\\ a_1, 1+a_1-a_2, a_1+a_2-n, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5\end{matrix};\right] \text{ and}$$

$$\frac{(1+a_3-a_2)_n(a_4+a_5)_n}{(1-a_1-a_2)_n(a_1+a_3+a_4+a_5)_n}\cdot {}_4F_3\left[\begin{matrix}a_1+a_3, 1-a_2-a_4, 1-a_2-a_5, -n\\ 1+a_3-a_2, 1+a_1-a_2, 1-a_4-a_5-n\end{matrix};\right]$$

$$= {}_4F_3\left[\begin{matrix}a_1+a_3, a_1+a_4, a_1+a_5, -n\\ 1+a_1-a_2, a_1+a_3+a_4+a_5, a_1+a_2-n\end{matrix};\right], \qquad (2.15.(1))$$

satisfy the same recurrence relation

If $n=0$ the ${}_7F_6$ becomes a reducible ${}_5F_4\left[\begin{matrix}2a_1, a_1+1, a_1+a_3, a_1+a_4, a_1+a_5\\ a_1, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5\end{matrix};\right]$ which we write as ${}_5F_4(a_1)$. If $n=1$ the ${}_7F_6$ is equal to

$$\sum_{r=0}^{\infty}\frac{(2a_1)_r(a_1+r)(a_1+a_3)_r(a_1+a_4)_r(a_1+a_5)_r}{r!\,a_1(1+a_1-a_3)_r(1+a_1-a_4)_r(1+a_1-a_5)_r}\cdot\frac{(a_1+a_2+r-1)(a_1-a_2+r+1)}{(a_1+a_2-1)(1+a_1-a_2)},$$

$$= \sum_{r=0}^{\infty}K_r\,\frac{(a_1+r)^2-(a_2-1)^2}{a_1^2-(a_2-1)^2}, \qquad (\text{say}),$$

$$= \sum_{r=0}^{\infty}K_r + \frac{1}{a_1^2-(a_2-1)^2}\sum_{r=1}^{\infty}r(r+2a_1)K_r,$$

$$= {}_5F_4(a_1) + \frac{1}{a_1^2-(a_2-1)^2}\cdot\frac{2a_1(2a_1+1)(a_1+1)(a_1+a_3)(a_1+a_4)(a_1+a_5)}{a_1(1+a_1-a_3)(1+a_1-a_4)(1+a_1-a_5)}\cdot$$

$$\cdot\sum_{r=0}^{\infty}\frac{(2a_1+2)_r(a_1+2)_r(1+a_1+a_3)_r(1+a_1+a_4)_r(1+a_1+a_5)_r}{r!\,(1+a_1)_r(2+a_1-a_3)_r(2+a_1-a_4)_r(2+a_1-a_5)_r},$$

$$= {}_5F_4(a_1) + \frac{(1+2a_1)(2+2a_1)(a_1+a_3)(a_1+a_4)(a_1+a_5)}{(1+a_1-a_3)(1+a_1-a_4)(1+a_1-a_5)(a_1-a_2+1)(a_1+a_2-1)}\,{}_5F_4(a+1).$$

Both the ${}_5F_4$'s are reducible, in fact

$${}_5F_4(a_1+1) = \frac{-(1+a_1-a_3)(1+a_1-a_4)(1+a_1-a_5)}{(a_1+a_3+a_4+a_5)(2+2a_1)(1+2a_1)}\,{}_5F_4(a_1),^{(1)}$$

so that when $n=1$ the ${}_7F_6$ is equal to

$${}_5F_4(a_1)\left\{1-\frac{(a_1+a_3)(a_1+a_4)(a_1+a_5)}{(1+a_1-a_2)(a_1+a_2-1)(a_1+a_3+a_4+a_5)}\right\},$$

$$= {}_5F_4(a_1).\,{}_4F_3\left[\begin{matrix}a_1+a_3, a_1+a_4, a_1+a_5, -1\\ 1+a_1-a_2, a_1+a_3+a_4+a_5, a_1+a_2-1\end{matrix};\right].$$

This completes the proof of Whipple's relation in the case where

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(1) tract. 4.4.(1).

the ${}_7F_6$ does not terminate

$$
{}_7F_6\left[\begin{matrix} 2a_1, a_1+1, a_1+a_2, 1+a_1-a_2+n, & a_1+a_3, a_1+a_4, a_1+a_5 \\ a_1, 1+a_1-a_2, a_1+a_2-n, & 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5 \end{matrix};\right]
$$

$$
= {}_5F_4\left[\begin{matrix} 2a_1, a_1+1, a_1+a_3, a_1+a_4, a_1+a_5 \\ a_1, 1+a_1-a_3, 1+a_1-a_4, 1+a_1-a_5 \end{matrix};\right] \cdot {}_4F_3\left[\begin{matrix} a_1+a_3, a_1+a_4, a_1+a_5, -n \\ 1+a_1-a_2, a_1+a_3+a_4+a_5, a_1+a_2-n \end{matrix};\right].
$$

Whipple's relation, Bailey's relation, and the transformation connecting two Saalschützian ${}_4F_3$, 2.15.(1). are typical of a large number of transformations which have been systematically worked out by Whipple and Bailey.[1] The natural tool to employ in dealing with such series of high order appears to be contour integrals of the Barnes type.[2]

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(1) tract. Chapter VII. also BAILEY. 1. WHIPPLE. 2.3.4.

(2) cf. for example WHIPPLE 2 and 3.

CHAPTER III

BASIC HYPERGEOMETRIC SERIES.

3.1. INTRODUCTORY REMARKS. We denote by

$$ {}_p\Phi_{p-1}\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_{p-1}\end{matrix};x\right], \text{ or simply } \Phi\left[\begin{matrix} a_r \\ b_s\end{matrix};x\right], $$

the generalised basic hypergeometric series

$$ \sum_{n=0}^{\infty} \frac{(a_1)_{q,n}(a_2)_{q,n}\cdots(a_p)_{q,n}}{(q)_{q,n}(b_1)_{q,n}\cdots(b_{p-1})_{q,n}}\, x^n \qquad , \quad |q|<1, \ |x|<1, $$

where $(a)_{q,n} = (1-a)(1-aq)\cdots(1-aq^{n-1})$, $(a)_{q,0} = 1$.

The possibility of any b_s being of the form q^{-N} where N is an integer is excluded; if any a_r is of this form then the series terminates.

When $q \to 1$ this series reduces to $F\left[\begin{matrix}\alpha_1,\dots \alpha_p \\ \beta_1,\dots \beta_{p-1}\end{matrix};x\right]$ where $a_r = q^{\alpha_r}$, $b_s = q^{\beta_s}$. It is also convergent for values of $|q|$ greater than unity provided $x < \left|\frac{q b_1 b_2 \cdots b_{p-1}}{a_1 a_2 \cdots a_p}\right|$.

Analogous to the transformations of ordinary hypergeometric series considered in the last chapter there are corresponding identities for basic series, the most important being

$$ (1) \quad {}_2\Phi_1\left[\begin{matrix} a, b; \frac{c}{ab} \\ c \end{matrix}\right] = \prod_{n=0}^{\infty} \frac{(1-\frac{cq^n}{a})(1-\frac{cq^n}{b})}{(1-cq^n)(1-\frac{cq^n}{ab})} = \frac{(\frac{c}{a})_{q,\infty}(\frac{c}{b})_{q,\infty}}{(c)_{q,\infty}(\frac{c}{ab})_{q,\infty}}, \ [1] $$

$$ |q|<1, \quad \left|\frac{c}{ab}\right| < 1. $$

This is the basic analogue of Gauss' theorem when $|q|<1$. For values of $|q|$ greater than unity the theorem takes the form,

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[1] Tract. 8.4.(3).

(2) $$ {}_2\Phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q\right] = \frac{(\frac{c}{a})_{q,\infty}(\frac{c}{b})_{q,\infty}}{(c)_{q,\infty}(\frac{c}{ab})_{q,\infty}}, \quad |q|>1, \left|\frac{c}{ab}\right|>1, $$

(3) $$ {}_3\Phi_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q\right] = \frac{(\frac{b_2}{a_1})_{q,N}(\frac{b_2}{a_2})_{q,N}}{(b_2)_{q,N}(\frac{b_2}{a_1a_2})_{q,N}}, \quad \frac{b_1b_2}{a_1a_2a_3} = q, \quad a_3 = q^{-N}, $$ [1]

(4) $$ {}_8\Phi_7\left[\begin{matrix} a_1^2, a_1q, -a_1q, a_1a_2, a_1a_3, a_1a_4, a_1a_5, a_1a_6 \\ a_1, -a_1, \frac{a_1q}{a_2}, \frac{a_1q}{a_3}, \frac{a_1q}{a_4}, \frac{a_1q}{a_5}, \frac{a_1q}{a_6} \end{matrix}; q\right] $$
$$ = \frac{(\frac{a_6}{a_1})_{q,N}(\frac{q}{a_2a_5})_{q,N}(\frac{q}{a_3a_5})_{q,N}(\frac{q}{a_4a_5})_{q,N}}{(\frac{a_1q}{a_5})_{q,N}(a_2a_6)_{q,N}(a_3a_6)_{q,N}(a_4a_6)_{q,N}}, $$

where $a_1a_2a_3a_4a_5a_6 = q$, $a_1a_6 = q^{-N}$,

together with an identity which may be stated as follows.

If $$ \phi(a_r, b_r) = \frac{(\frac{a_1q}{b_3})_{q,N}(\frac{a_1q}{b_4})_{q,N}}{(a_1^2q)_{q,N}(\frac{q}{b_3b_4})_{q,N}}\; {}_{10}\Phi_9\left[\begin{matrix} a_1a_r, & a_1b_r, & a_1q, & -a_1q \\ \frac{a_1q}{a_r}, & \frac{a_1q}{b_r}, & a_1, & -a_1 \end{matrix}; q\right], $$

$r = 1, 2, 3, 4,$ then

(5) $$ \phi(a_r\lambda, \frac{b_r}{\lambda}) = \phi(\frac{a_r}{\lambda}, b_r\lambda), $$

subject to the conditions $a_1a_2a_3a_4 = b_1b_2b_3b_4 = q$, $a_1b_1 = q^{-N}$. [2]

Equation (3) is the basic form of Saalschütz' theorem, the normal Saalschützian condition being replaced by the condition that the product of the denominator parameters is q times the product of the numerator parameters. Equations (4) and (5) are the basic forms of Dougall's identity and Bailey's identity, the basic 'well-poised' condition being that the numerator and denominator parameters can be arranged in pairs which have the same product, this product being q times the remaining numerator parameter.

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[1] Tract. 8.4.(1).   [2] Tract. 8.5.(1).

In this chapter we adapt the methods of Chapter II to give simple direct proofs of certain identities concerning reducible series. Exactly as in the previous chapter the method can be used, but with a similar loss of elegance, to verify the various identities between non-reducible series such as Bailey's identity above. Here we will merely indicate how such a verification is achieved. The chapter concludes with an account of basic series in the basic number notation developed by F. H. Jackson.

3.2. THE OPERATOR Q.

We define an operator Q by the following relations

(1) $$Q.f(x) = f(qx)\ , \quad Q.a = a\ , \quad a \text{ independent of } x.$$

Symbolically $Q \equiv q^{\delta}$, since

$$\begin{aligned} q^{\delta}x^n &= \sum \frac{\delta^r(\log q)^r}{r!}x^n\,, \\ &= \sum \frac{n^r(\log q)^r}{r!}x^n\,, \qquad f(\delta)x^n = f(n)x^n, \\ &= q^n x^n. \end{aligned}$$

This operator has been considered by many writers. It is for example the η of Rogers,[1] and the q^{θ} of Jackson's later papers.[2] The symbol Q adopted here belongs to one of Jacksons earlier papers.[3]

Corresponding to the properties of the δ-operator,

$$f(\delta)x^n = f(n)x^n\,, \quad f(\delta)x^n y = x^n f(\delta+n)y,$$

we may readily obtain

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[1] ROGERS and RAMANUJAN. 1. [2] e.g. JACKSON. F.H. 10.

[3] JACKSON. F.H. 5.

(2) $$f(Q)x^n = f(q^n)x^n,$$

(3) $$f(Q)x^n y = x^n f(q^n Q)y,$$

where $f(Q)$ is a polynomial in Q.

The following relations are also easily verified,

(4) $$(1-Q)\,\Phi\left[\begin{matrix}a_r\\ b_s\end{matrix};x\right] = \prod\frac{(1-a_r)}{(1-b_s)}\,x\,\Phi\left[\begin{matrix}qa_r\\ qb_s\end{matrix};x\right],$$

(5) $$(1-a_r Q)\,\Phi\left[\begin{matrix}a_r\\ b_s\end{matrix};x\right] = (1-a_r)\,\Phi\left[\begin{matrix}qa_r\\ b_s\end{matrix};x\right],$$

(6) $$\left(1-\frac{b_s}{q}Q\right)\Phi\left[\begin{matrix}a_r\\ b_s\end{matrix};x\right] = \left(1-\frac{b_s}{q}\right)\Phi\left[\begin{matrix}a_r\\ \frac{b_s}{q}\end{matrix};x\right],$$

where in (4) every parameter is multiplied by q, whilst in (5) and (6) only the parameter specified is multiplied or divided by q.

From these relations we may deduce that $\Phi\left[\begin{matrix}a_r\\ b_s\end{matrix};x\right]$ is a solution of the 'q-difference equation'

(7) $$\left[(1-Q)\prod_s\left(1-\frac{b_s}{q}Q\right) - x\prod_r(1-a_r Q)\right]\Phi = 0.$$

Equations (4), (5), (6) and (7) above correspond for basic series to 2.2. (1), (2), (3) and (5) for ordinary series, and enable us to obtain identities for basic reducible series exactly as, in the last chapter, we obtained identities for ordinary reducible series.

For example the series ${}_1\Phi_0\left[\begin{matrix}a\\ \end{matrix};x\right]$ is annihilated by the operator $[(1-Q)-x(1-aQ)]$, which is linear in Q and may be rewritten in the form

$$[(1-x)-(1-ax)Q]$$

so that performing on ${}_1\Phi_0\left[\begin{matrix}a\\ \end{matrix};x\right]$ the operations indicated we obtain

$$ {}_1\Phi_0\left[{a:\atop};x\right] = \frac{(1-ax)}{(1-x)}\,{}_1\Phi_0\left[{a\atop};qx\right] $$

$$ = \frac{(ax)_{q,n}}{(x)_{q,n}}\,{}_1\Phi_0\left[{a\atop};q^nx\right]. $$

If $|q|<1$, ${}_1\Phi_0\left[{a\atop};q^nx\right] \to 1$ as $n\to\infty$ and therefore

(8) $$ {}_1\Phi_0\left[{a\atop};x\right] = \frac{(ax)_{q,\infty}}{(x)_{q,\infty}}, \qquad |q|<1\ .^{(1)} $$

If $|q|>1$, then from above we deduce that

$$ {}_1\Phi_0\left[{a\atop};x\right] = \frac{(1-\frac{x}{q})}{(1-\frac{ax}{q})}\,{}_1\Phi_0\left[{a\atop};\frac{x}{q}\right] $$

$$ = \frac{(1-\frac{x}{q})(1-\frac{x}{q^2})\cdots(1-\frac{x}{q^n})}{(1-\frac{ax}{q})(1-\frac{ax}{q^2})\cdots(1-\frac{ax}{q^n})}\,{}_1\Phi_0\left[{a\atop};\frac{x}{q^n}\right], $$

(9) i.e. $$ {}_1\Phi_0\left[{a\atop};x\right] = \prod_{n=1}^{\infty}\frac{(1-\frac{x}{q^n})}{(1-\frac{ax}{q^n})}, \qquad |q|>1 $$

since limit ${}_1\Phi_0\left[a;\frac{x}{q^n}\right] = 1$.

3.3. THE BASIC ANALOGUES OF GAUSS' AND SAALSCHÜTZ' THEOREMS.

For series of higher order we give special values to x, and place restrictions on the parameters so that the operator annihilating the series becomes linear in Q. If the series terminates we may give to x and the parameters any values we please; if the series does not terminate the values given must be consistent with the convergence of all the series implied in the operational relation.

To prove Gauss' theorem we consider ${}_2\Phi_1\left[{a,b\atop cq};x\right]$ which is a 'solution' of the q-difference equation

(1) $$ [(1-Q)(1-cQ) - x(1-aQ)(1-bQ)]\,\Phi = 0. $$

---

(1) tract. 8.2.(4).

Setting $x = \frac{c}{ab}$, the coefficient of Q^2 in the equation vanishes, and the operator may be written

(2) $$[A(1-cQ) + BQ].$$

Evaluating A and B between (1) and (2) gives

$$A = \left(1-\frac{c}{ab}\right), \qquad B = \frac{-c^2}{ab}\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right),$$

and using 3.2.(6) we obtain

$$_2\Phi_1\left[\begin{matrix}a, b\\ c\end{matrix}; \frac{c}{ab}\right] = \frac{(1-\frac{c}{a})(1-\frac{c}{b})}{(1-\frac{c}{ab})(1-c)}\, {}_2\Phi_1\left[\begin{matrix}a, b\\ cq\end{matrix}\ \frac{cq}{ab}\right].$$

We may repeat this process and since $_2\Phi_1\left[\begin{matrix}a, b\\ cq^n\end{matrix}; \frac{cq^n}{ab}\right] \to 1$, for $|q|<1$ we have eventually

$$_2\Phi_1\left[\begin{matrix}a, b\\ c\end{matrix}; \frac{c}{ab}\right] = \frac{(\frac{c}{a})_{q,\infty}(\frac{c}{b})_{q,\infty}}{(\frac{c}{ab})_{q,\infty}(c)_{q,\infty}}, \qquad |q|<1.$$

If $|q|>1$ we assume that $|\frac{c}{ab}|$ is sufficiently large and set $x=1$ in (1). The operator may now be written in the form

$$[A(1-cQ) + B]\,Q,$$

and evaluation of A and B leads to the result

$$_2\Phi_1\left[\begin{matrix}a, b\\ c\end{matrix}; q\right] = \frac{(1-\frac{c}{a})(1-\frac{c}{b})}{(1-\frac{c}{ab})(1-c)}\, {}_2\Phi_1\left[\begin{matrix}a, b\\ cq\end{matrix}; q\right],$$

and eventually since $_2\Phi_1\left[\begin{matrix}a, b\\ cq^N\end{matrix}; q\right] \to 1$ we obtain the alternative form

$$_2\Phi_1\left[\begin{matrix}a, b\\ c\end{matrix}; q\right] = \frac{(\frac{c}{a})_{q,\infty}(\frac{c}{b})_{q,\infty}}{(\frac{c}{ab})_{q,\infty}(c)_{q,\infty}}, \qquad |q|>1.$$

To prove Saalschütz theorem we consider the operator corresponding to the series

$$_3\Phi_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1 q, b_2 q\end{matrix}; 1\right] \quad \text{where } b_1 b_2 = a_1 a_2 a_3, \ a_3 = q^{-N}.$$

(3) This operator $[(1-Q)(1-b_1Q)(1-b_2Q)-(1-a_1Q)(1-a_2Q)(1-a_3Q)]$ may be written

$$[A(1-b_2Q)-B(1-a_3Q)]\,Q$$

where $A = a_3(1-\frac{1}{a_3})(1-\frac{b_1}{a_3})$; $B = b_2(1-\frac{a_1}{b_2})(1-\frac{a_2}{b_2})$.

Using 3.2. (5) and (6) we find that

$${}_3\Phi_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1q, b_2\end{matrix}; q\right] = \frac{(1-\frac{b_2}{a_1})(1-\frac{b_2}{a_2})}{(1-b_2)(1-\frac{a_3}{b_1})}\, {}_3\Phi_2\left[\begin{matrix}a_1, a_2, a_3q\\ b_1q, b_2q\end{matrix}; q\right]$$

where both series are now Saalschützian. Repeating the process N times, the final series on the right hand side reduces to unity, and writing $\frac{b_1}{q}$ for b_1 we have Saalschütz' theorem

$${}_3\Phi_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; q\right] = \frac{(\frac{b_2}{a_1})_{q,N}(\frac{b_2}{a_2})_{q,N}}{(b_2)_{q,N}(\frac{b_2}{a_1a_2})_{q,N}}, \quad \frac{b_1b_2}{a_1a_2a_3} = q, \quad a_3 = q^{-N}.$$

If we let $N \to \infty$ in this relation we obtain Gauss' theorem either 3.1 (1) or 3.1 (2) according to whether $|q|$ is less than or greater than unity.

Having obtained the reduced expressions for the ${}_2\Phi_1$, and ${}_3\Phi_2$, we may rewrite the corresponding operators in various different ways in order to obtain pseudo-reducible series exactly as in 2.11.

Thus for example the operator (3) may be written either as

(4) $$[A(1-b_2Q)-B]\,Q, \quad \text{or}$$

(5) $$[A(1-b_2Q)-BQ]\,Q, \quad \text{or}$$

(6) $$[A(1-b_2Q)-B(1-cQ)]\,Q, \quad \text{or}$$

(7) $$[AQ-BQ^2].$$

From (4), (5) and (6) we obtain the following identities,

$$(8)\quad {}_3\Phi_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1q,b_2q\end{matrix};q\right] = \frac{A(1-b_2)}{B}\,{}_3\Phi_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1q,b_2\end{matrix};q\right],\quad b_1b_2=a_1a_2a_3,\ a_3=q^{-N};$$

$$= \frac{A(1-b_2)\left(\frac{b_2}{a_1}\right)_{q,N}\left(\frac{b_2}{a_2}\right)_{q,N}}{B(b_2)_{q,N}\left(\frac{b_2}{a_1a_2}\right)_{q,N}},$$

where $A = \dfrac{a_1a_2+a_2a_3+a_3a_1-b_1-b_2-b_1b_2}{b_2}$, $B = b_2\left(1-\frac{a_1}{b_2}\right)\left(1-\frac{a_2}{b_2}\right)\left(1-\frac{a_3}{b_2}\right).$

$$(9)\quad {}_3\Phi_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1q,b_2q\end{matrix};q^2\right] = \frac{A(1-b_2)\left(\frac{b_2}{a_1}\right)_{q,N}\left(\frac{b_2}{a_2}\right)_{q,N}}{B(b_2)_{q,N}\left(\frac{b_2}{a_1a_2}\right)_{q,N}},\quad b_1b_2=a_1a_2a_3,\ a_3=q^{-N}$$

where $A = a_1+a_2+a_3-1-b_1-b_2$; $B = b_2^2\left(1-\frac{a_1}{b_2}\right)\left(1-\frac{a_2}{b_2}\right)\left(1-\frac{a_3}{b_2}\right).$

$$(10)\quad {}_4\Phi_3\left[\begin{matrix}a_1,a_2,a_3,\ cq\\ b_1q,b_2q,\ c\end{matrix};q\right] = \frac{A(1-b_2)\left(\frac{b_2}{a_1}\right)_{q,N}\left(\frac{b_2}{a_2}\right)_{q,N}}{B(1-c)(b_2)_{q,N}\left(\frac{b_2}{a_1a_2}\right)_{q,N}},\quad b_1b_2=a_1a_2a_3,\ a_3=q^{-N}$$

where this time $A\left(1-\frac{b_2}{c}\right)\frac{1}{c} = \left[\left(1-\frac{1}{c}\right)\left(1-\frac{b_1}{c}\right)\left(1-\frac{b_2}{c}\right)-\left(1-\frac{a_1}{c}\right)\left(1-\frac{a_2}{c}\right)\left(1-\frac{a_3}{c}\right)\right]$

and $B\left(1-\frac{c}{b_2}\right)\frac{1}{b_2} = \left[\left(1-\frac{a_1}{b_2}\right)\left(1-\frac{a_2}{b_2}\right)\left(1-\frac{a_3}{b_2}\right)\right].$

The operator written in the form (7) leads to a relation between ${}_3\Phi_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1q,b_2q\end{matrix};q\right]$ and ${}_3\Phi_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1q,b_2q\end{matrix};q^2\right]$ which is an immediate deduction from (8) and (9).

Results of a similar nature may be obtained by rearranging the Gaussian operator (1), $\left[(1-Q)(1-cQ)-\frac{c}{ab}(1-aQ)(1-bQ)\right]$, in either of the forms

$$[A-BQ],$$

$$[A-B(1-dQ)].$$

3.4. 'WELL-POISED' BASIC SERIES. In dealing with 'well-poised' basic series it is advantageous to choose parameters in the form indicated by 3.1.(4). Then the series $\Phi\left[\begin{matrix}a_1^2,\ a_1a_r\\ \frac{a_1q}{a_r}\end{matrix};x\right]$ satisfies the q-difference equation

$$(1)\quad \left[\prod_{r=1}^{n}\left(1-\frac{a_1}{a_r}Q\right)-x\prod_{r=1}^{n}\left(1-a_1a_rQ\right)\right]\Phi = 0.$$

Also

(2) $$(1-a_1a_rQ)(1-\tfrac{a_1}{a_r}Q)\,\Phi\left[\begin{matrix}a_1^2, a_1a_r \\ \frac{a_1q}{a_r}\end{matrix};x\right] = (1-a_1a_r)(1-\tfrac{a_1}{a_r})\,\Phi\left[\begin{matrix}a_1^2, a_1a_rq \\ \frac{a_1}{a_r}\end{matrix};x\right],$$

the effect being to multiply a_r by q both where it occurs in a numerator and a denominator parameter, whilst preserving the 'well-poised' condition, and

$$(1\pm a_1Q)\,\Phi\left[\begin{matrix}a_1^2, a_1a_r \\ \frac{a_1q}{a_r}\end{matrix};x\right] = (1\pm a_1)\,\Phi\left[\begin{matrix}a_1^2, \mp a_1q, a_1a_r \\ \mp a_1, \frac{a_1q}{a_r}\end{matrix};x\right],$$

leading to

(3) $$(1-a_1^2Q^2)\,\Phi\left[\begin{matrix}a_1^2, a_1a_r \\ \frac{a_1q}{a_r}\end{matrix};x\right] = (1-a_1^2)\,\Phi\left[\begin{matrix}a_1^2, a_1q, -a_1q, a_1a_r \\ a_1, -a_1, \frac{a_1q}{a_r}\end{matrix};x\right]$$

introducing two further numerator and denominator parameters.

3.5. THE BASIC DOUGALL THEOREM.

Most of the known cases of reducible 'well-poised' basic series are special cases of the basic analogue of Dougall's theorem which we now prove.

${}_6\Phi_5\left[\begin{matrix}a_1^2, a_1a_r \\ \frac{a_1q}{a_r}\end{matrix};x\right]$ is a solution of the q-difference equation

$$\left[\prod_{r=1}^{6}(1-\tfrac{a_1}{a_r}Q) - x\prod_{r=1}^{6}(1-a_1a_rQ)\right]\Phi = 0.$$

Let $a_1a_6 = q^{-N}$, and further let

(1) $$a_1a_2a_3a_4a_5a_6 = 1, \quad \text{i.e. } \sigma_6^6 = 1,$$

where σ_r^6 is the r^{th} elementary symmetric multiplier of $a_1, \ldots, a_6$.

We notice that because of (1)

$$\sigma_{-r}^6 = \sigma_{6-r}^6,$$

and therefore setting $x=1$ in the operator above, we have

$$\left[\prod(1-\tfrac{a_1}{a_r}Q) - \prod(1-a_1a_rQ)\right]$$
$$= \left[\sum_{r=0}^{6}(-)^r\sigma_{-r}^6(a_1Q)^r - \sum_{r=0}^{6}(-)^r\sigma_r^6(a_1Q)^r\right],$$

$$= (\sigma_1 - \sigma_5)(a_1 Q - a_1^5 Q^5) - (\sigma_2 - \sigma_4)(a_1^2 Q^2 - a_1^4 Q^4)$$

$$= a_1 Q (1 - a_1^2 Q^2) \left[A(1 + a_1^2 Q^2) + B a_1 Q \right],$$

which may be written

$$(2) \quad \left[\prod_{r=1}^{6} \left(1 - \frac{a_1}{a_r} Q\right) - \prod_{r=1}^{6} (1 - a_1 a_r Q) \right]$$

$$\equiv a_1 Q (1 - a_1^2 Q^2) \left[A'(1 - a_1 a_6 Q)\left(1 - \frac{a_1}{a_6} Q\right) + B'(1 - a_1 a_5 Q)\left(1 - \frac{a_1}{a_5} Q\right) \right].$$

Setting $Q = \frac{a_5}{a_1}, \frac{a_6}{a_1}$, in turn in (2) we find that

$$A' = \frac{-(1 - a_1 a_5)(1 - a_2 a_5)(1 - a_3 a_5)(1 - a_4 a_5)}{a_5 \left(1 - \frac{a_5}{a_6}\right)},$$

$$B' = \frac{-(1 - a_1 a_6)(1 - a_2 a_6)(1 - a_3 a_6)(1 - a_4 a_6)}{a_6 \left(1 - \frac{a_6}{a_5}\right)}.$$

Performing on Φ the operations indicated on the right hand side of (2) using 3.4.(2) and (3), we find on rearranging that if

$$\phi(a_5, a_6, q) = {}_8\Phi_7 \left[\begin{matrix} a_1^2, a_1 q, -a_1 q, \frac{a_1 a_r}{\ } \\ a_1, -a_1, \frac{a_1 q}{a_r} \end{matrix} ; q \right], \qquad r = 2, 3, 4, 5, 6,$$

then
$$\phi(a_5 q, a_6, q) = \frac{\left(1 - \frac{a_6}{a_1}\right) \prod^{r=2,3,4} \left(1 - \frac{1}{a_r a_5}\right)}{\left(1 - \frac{a_1}{a_5}\right) \prod^{r=2,3,4} (1 - a_r a_6)} \phi(a_5, a_6 q, q).$$

Repeating the process N times and noting that $\phi(a_5, a_6 q^N, q) = 1$, we have finally, writing $\frac{a_5}{q}$ for a_5

$${}_8\Phi_7 \left[\begin{matrix} a_1^2, a_1 q, -a_1 q, a_1 a_r \\ a_1, -a_1, \frac{a_1 q}{a_r} \end{matrix} ; q \right] = \frac{\left(\frac{a_6}{a_1}\right)_{q,N}}{\left(\frac{a_1 q}{a_5}\right)_{q,N}} \prod^{r=2,3,4} \frac{\left(\frac{q}{a_r a_5}\right)_{q,N}}{(a_r a_6)_{q,N}},$$

$$\text{where } a_1 a_6 = q^{-N}, \quad \sigma_6^6 = q,$$

which is the basic analogue of Dougall's theorem.

3.6. IDENTITIES RELATED TO DOUGALL'S THEOREM.

Results analogous to Dixon's theorem, Kummer's theorem etc., are obtained by letting $N \to \infty$ in Dougall's theorem, and then giving special values to the parameters. Such identities are well known[1] and we do not reproduce them here. We will however obtain from Dougall's theorem, written in the form in which we have given it, an identity due to Jackson.[2]

The left-hand side of Dougall's identity is

$$\sum_{m=0}^{N} \frac{(a_1^2)_{q,m}\,(q^{-N})_{q,m}\,(1-a_1^2q^{2m})}{(q)_{q,m}\,(a_1^2q^{N+1})_{q,m}\,(1-a_1^2)}\, q^m \prod \frac{(a_1a_r)_{q,m}}{\left(\frac{a_1q}{a_r}\right)_{q,m}}, \quad r = 2,3,4,5.$$

In this let $a_1 \to q^{-N}$. Then

$$(1) \qquad \frac{(q^{-N})_{q,m}\,(1-a_1^2q^{2m})}{(a_1^2q^{N+1})_{q,m}\,(1-a_1^2)} = \frac{(1+q^{m-N})}{(1+q^{-N})}, \quad m \neq N$$

$$= (1+q^{-N})^{-1}, \quad m = N.$$

Also $\dfrac{(a_1^2)_{q,m}}{(q)_{q,m}} \to \dfrac{(q^{-2N})_{q,m}}{(q)_{q,m}} = \dfrac{(q^{-2N})_{q,2N-m}}{(q)_{q,2N-m}} \cdot q^{(N-m)(2N+1)}$,

and $\dfrac{(a_1a_r)_{q,m}}{\left(\frac{a_1q}{a_r}\right)_{q,m}} \to \dfrac{(q^{-N}a_r)_{q,m}}{\left(\frac{q^{-N+1}}{a_r}\right)_{q,m}} = \dfrac{(q^{-N}a_r)_{q,2N-m}}{\left(\frac{q^{-N+1}}{a_r}\right)_{q,2N-m}} \cdot (a_r)^{2m-2N}\, q^{(N-m)}$,

so that combining these results

$$(2) \qquad \frac{(q^{-2N})_{q,m}}{(q)_{q,m}} \prod \frac{(q^{-N}a_r)_{q,m}}{\left(\frac{q^{-N+1}}{a_r}\right)_{q,m}} = \frac{(q^{-2N})_{q,2N-m}}{(q)_{q,2N-m}} \prod \frac{(q^{-N}a_r)_{q,2N-m}}{\left(\frac{q^{-N+1}}{a_r}\right)_{q,2N-m}} \cdot q^{3N-3m}.$$

Thus splitting each term of the series into two parts, and combining the left-hand side of (2) with $\dfrac{q^{m-N}}{(1+q^{-N})}$ of (1) and the right-hand side

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[1] vide for example BAILEY. 4. These identities take slightly different forms for $|q|<1$ and $|q|>1$.

[2] JACKSON. F. H. 11. see also BAILEY. 4.

of (2) with $\overline{\dfrac{1}{(1+q^{-N})}}$ of (1) we obtain

$$(3)\quad {}_5\Phi_4\left[\begin{matrix} q^{-2N}, & q^{-N}a_r \\ & \frac{q^{-N+1}}{a_r} \end{matrix}; q^2\right] = \frac{(1+q^{N})(q^{N})_{q,N}}{\left(\frac{q^{-N+1}}{a_5}\right)_{q,N}} \prod \frac{\left(\frac{q}{a_r a_5}\right)_{q,N}}{(a_r)_{q,N}}, \quad r=2,3,4,$$

where $a_2a_3a_4a_5 = q^{N+1}$.

We will also prove a result analogous to 2.10. (1).

Returning to the operator 3.5. (2) we have

$$\left[\prod_{r=1}^{6}\left(1-\frac{a_1}{a_r}Q\right) - \prod_{r=1}^{6}(1-a_1a_rQ)\right]$$

$$= a_1Q(1-a_1^2Q^2)\left[A(1-a_1a_5Q)\left(1-\frac{a_1}{a_5}Q\right) + Ba_1Q\right].$$

Setting $Q = \frac{a_5}{a_1}$ we find

$$B = \frac{-\prod(1-a_ra_5)}{a_5^2} \qquad r = 1,2,3,4,6,$$

whilst equating the coefficients of $-a_1^5Q^5$

$$A = (\sigma_1^6 - \sigma_5^6).$$

We thus obtain a relation between two series of the type ${}_8\Phi_7$, one of which is reducible by Dougall's theorem, so that eventually we have

$$(4)\quad {}_8\Phi_7\left[\begin{matrix} a_1^2, a_1q, -a_1q, a_1a_r \\ a_1, -a_1, \frac{a_1q}{a_r} \end{matrix}; q^2\right]$$

$$= \frac{(\sigma_1^6-\sigma_5^6)\,a_5^2\left(1-\frac{a_1}{a_5}\right)}{a_1(1-a_2a_5)(1-a_3a_5)(1-a_4a_5)(1-a_6a_5)} \frac{\left(\frac{a_6}{a_1}\right)_{q,N}}{\left(\frac{a_1}{a_5}\right)_{q,N}} \prod^{r=2,3,4} \frac{\left(\frac{1}{a_ra_5}\right)_{q,N}}{(a_ra_6)_{q,N}},$$

subject to the conditions $a_1a_6 = q^{-N}$, $\sigma_6^6 = 1$.

3.7. THE BASIC ANALOGUE OF BAILEY'S IDENTITY.

Equation 3.1. (4), the basic analogue of Bailey's identity, may be verified in exactly the same way as we verified the original identity in Chapter II. We begin by considering the operator

3.8. THE BASIC NUMBER NOTATION OF F. H. JACKSON. In a number of papers[1], F. H. Jackson has succeeded in developing a notation which presents the theory of basic hypergeometric series in a manner which, in appearance, is identical with large sections of the theory of ordinary hypergeometric series. The notation has certain serious drawbacks, principally arising from the fact that the basic numbers used do not obey any simple laws corresponding to multiplication, so that, for example, the method of proof of Dougall's theorem given earlier in this chapter could not be transferred neatly into Jacksons notation. Nevertheless it seems worth while to collect together that part of Jacksons work relevant to this subject.

The basic number $[\alpha]$ was originally defined to be the principal value of the expression $\left(\frac{1-q^{\alpha}}{1-q}\right)$. This definition has the advantage that as $q \to 1$, $[\alpha]$ tends to the ordinary number α. In later papers[2] however the denominator factor is omitted, and we will follow this practice. The basic number $[\alpha]$ is therefore defined by

$$[\alpha] = 1-q^{\alpha}.$$

An equation such as $[\alpha] = 0$ has a doubly infinite system of roots $\alpha = \frac{2r\pi i}{\log q + 2\pi s i}$ $(r = 0,1,2,\cdots ; s=0,1,2,\cdots)$. Where such equations arise we always select the principal value for α, in this case zero.

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[1] The bibliography lists a selection of these.

Some of the definitions which follow have a meaning for values of $|q| > 1$, and some of the results proved are true for such values of q with perhaps slight modifications. In what follows however we restrict ourselves to $|q| < 1$.

If n is an integer it is easy to show that

$$(1-x)^{(n)} = (1-x)(1-qx)\cdots(1-q^{n-1}x)$$

$$= 1 - \frac{[n]}{[1]}x + q\frac{[n][n-1]}{[2]!}x^2 - \cdots + (-)^r q^{\frac{1}{2}r(r-1)}\frac{[n]\cdots[n-r+1]}{[r]!}x^r + \cdots + (-)^n q^{\frac{1}{2}n(n-1)}x^n,$$

where $[r]! = [1][2]\cdots[r]$, the formal product.

For non-integral α, consider the series

$$1 - \frac{[\alpha]}{[1]}x + \cdots + (-)^r q^{\frac{1}{2}r(r-1)}\frac{[\alpha][\alpha-1]\cdots[\alpha-r+1]}{[r]!}x^r + \cdots .$$

Setting $a = q^\alpha$ this expression becomes

$$1 + \frac{(1-\frac{1}{a})}{(1-q)}ax + \frac{(1-\frac{1}{a})(1-\frac{q}{a})}{(1-q)(1-q^2)}a^2x^2 + \cdots$$

$$= {}_1\Phi_0\left[\tfrac{1}{a}; ax\right]$$ which by 3.2.(8) equals

$$\prod_{n=0}^{\infty}\frac{(1-q^n x)}{(1-q^n ax)} \qquad |q| < 1. \tag{1}$$

Defining $(1-x)^{(\alpha)}$ as the expression (1) for non-integral α, we obtain a basic analogue of the binomial theorem, namely

$$(1-x)^{(\alpha)} = \prod_{n=0}^{\infty}\frac{(1-q^n x)}{(1-q^n ax)} = \sum_{r=0}^{\infty}(-)^r\binom{[\alpha]}{[r]}q^{\frac{1}{2}r(r-1)}x^r, \tag{2}$$

where $\binom{[\alpha]}{[r]}$ is the basic binomial coefficient $\dfrac{[\alpha]\cdots[\alpha-r+1]}{[r]!}$

The series in (2) is convergent for $|q| < 1$, provided that $|x| < q^{-\alpha}$.

We also note that

$$(3) \qquad F_q\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma]\end{matrix}; q^{\gamma-\alpha-\beta}\right] = \lim_{n\to\infty} \frac{[\gamma-\alpha]_n[\gamma-\beta]_n}{[\gamma]_n[\gamma-\alpha-\beta]_n},$$

$$= \frac{\Gamma_q(\gamma)\Gamma_q(\gamma-\alpha-\beta)}{\Gamma_q(\gamma-\alpha)\Gamma_q(\gamma-\beta)},$$

identical in appearance with Gauss' theorem except that the argument is $q^{\gamma-\alpha-\beta}$. This relation is true provided $|q^{\gamma-\alpha-\beta}|<1$.

3.10. A BASIC OPERATOR ANALAGOUS TO δ.

The operator

$$[\delta] \equiv (1-q^{\delta})$$

has the following properties in imitation of the ordinary operator δ.

$$(1) \qquad [\delta]x^n = [n]x^n;$$

$$(2) \qquad [\delta]x^n y = x^n[\delta+n]y.$$

Equations (4), (5), (6) of 3.2. may be written

$$(3) \qquad [\delta]F\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma]\end{matrix};x\right] = \frac{[\alpha][\beta]}{[\gamma]}x F\left[\begin{matrix}[\alpha+1],[\beta+1]\\ [\gamma+1]\end{matrix};x\right],$$

$$(4) \qquad [\delta+\alpha]F\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma]\end{matrix};x\right] = [\alpha]F\left[\begin{matrix}[\alpha+1],[\beta]\\ [\gamma]\end{matrix};x\right],$$

$$(5) \qquad [\delta+\gamma-1]F\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma]\end{matrix};x\right] = [\gamma-1]F\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma-1]\end{matrix};x\right],$$

leading to the q-difference equation

$$(6) \qquad \{[\delta][\delta+\gamma-1] - x[\delta+\alpha][\delta+\beta]\}y=0,$$

satisfied by $F\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma]\end{matrix};x\right]$.

To a limited extent we may transform equation (6) just as we transform the ordinary hypergeometric equation. Thus setting $y=x^{1-\gamma}z$ and using relation (2) above, the equation becomes

$$\{[\delta][\delta+1-\gamma] - x[\delta+1+\alpha-\gamma][\delta+1+\beta-\gamma]\}z = 0$$

and hence another 'solution' of (6) is

$$x^{1-\gamma} F\left[\begin{matrix}[1+\alpha-\gamma],[1+\beta-\gamma]\\ [2-\gamma]\end{matrix};x\right].$$

If in (6) we set $x=\frac{1}{x}$, we have to replace δ by $-\delta$ and the equation becomes

$$\{x[-\delta][-\delta+\gamma-1]-[-\delta+\alpha][-\delta+\beta]\}y=0.$$

Since $[-\alpha] = (-)q^{-\alpha}[\alpha]$, this may be written

$$\{[\delta-\alpha][\delta-\beta]-xq^{1+\gamma-\alpha-\beta}[\delta][\delta-\gamma+1]\}y=0,$$

and we see that (6) also has the solutions

$$x^{-\alpha} F\left[\begin{matrix}[\alpha],[1+\alpha-\gamma]\\ [1+\alpha-\beta]\end{matrix};\frac{q^{1+\gamma-\alpha-\beta}}{x}\right],$$

$$x^{-\beta} F\left[\begin{matrix}[\beta],[1+\beta-\gamma]\\ [1+\beta-\alpha]\end{matrix};\frac{q^{1+\gamma-\alpha-\beta}}{x}\right].$$

Jackson also quotes solutions of the q-difference equation in powers of $(1-x)^{(r)}$, $\frac{x^r}{(1-q^\lambda x)^{(r)}}$ etc., corresponding to the twenty four solutions of the hypergeometric equation, solutions which we do not reproduce here,[1] except to note the transformation

$$F\left[\begin{matrix}[\alpha],[\beta]\\ [\gamma]\end{matrix};x\right] = (1-q^{\alpha+\beta-\gamma}x)^{(\gamma-\alpha-\beta)} F\left[\begin{matrix}[\gamma-\alpha],[\gamma-\beta]\\ [\gamma]\end{matrix};q^{\alpha+\beta-\gamma}x\right].^{(2)}$$

This relation may be verified by equating coefficients of x^n on each side using the basic analogue of Saalschütz theorem.

Finally we may also obtain the two term Thomae relations

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(1) Jackson. F. H. 1Q pp.16-17. A number of the 'solutions' given here do not appear to satisfy Jackson's equation.

(2) c.f. tract 8.4.(2).

$$ {}_3F_2\left[\begin{matrix}[\alpha_1][\alpha_2],[\alpha_3] \\ [\beta_1],[\beta_2]\end{matrix};q^s\right] = \frac{\Gamma_q(\beta_1)\Gamma_q(\beta_2)\Gamma_q(s)}{\Gamma_q(\alpha_1)\Gamma_q(s+\alpha_2)\Gamma_q(s+\alpha_3)}\,{}_3F_2\left[\begin{matrix}[\beta_1-\alpha_1],[\beta_2-\alpha_1],[s] \\ [s+\alpha_2],[s+\alpha_3]\end{matrix};q^{\alpha_1}\right], $$

$$ s = \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3, $$

and

$$ {}_3F_2\left[\begin{matrix}[\alpha_1],[\alpha_2],[\alpha_3] \\ [\beta_1],[\beta_2]\end{matrix};q^s\right] = \frac{\Gamma_q(\beta_2)\,\Gamma_q(s)}{\Gamma_q(\beta_2-\alpha_3)\Gamma_q(s+\alpha_3)}\,{}_3F_2\left[\begin{matrix}[\beta_1-\alpha_1],[\beta_1-\alpha_2],[\alpha_3] \\ [\beta_1],[s+\alpha_3]\end{matrix};q^{\beta_2-\alpha_3}\right]. $$

The first of these is obtained by following step by step the argument of 1.7. The second is merely another application of the transformation giving the first.

By considering the operator inverse to $\frac{1}{x}[\delta]$, Jackson has also developed a theory of 'q-integration' and has obtained 'q-integrals' analogous to the ordinary integral representations of the hypergeometric function, the gamma function, the beta function and other functions of mathematics.[1] With the processes of 'q-differentiation' and 'q-integration' he has, within limits, created the machinery with transfer many sections of ordinary function theory into the equivalent basic theory.

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(1) JACKSON. F.H. 1.2.3.8. etc.

CHAPTER. IV.

THE SYSTEM OF DIFFERENTIAL EQUATIONS SATISFIED BY APPELL'S FUNCTION $F_{(1)}$.

4.1. INTRODUCTION. Appell's hypergeometric function of two variables $F_{(1)}$ is defined by the series

$$F_{(1)}\left[\begin{matrix}a; & b, b'; x, y\\ c; & \end{matrix}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n,$$

which converges if $|x|<1$, $|y|<1$.[1]

If $\delta \equiv x\frac{d}{dx}$, $\delta' \equiv y\frac{d}{dy}$ then we may easily verify relations of which the following are typical

$$\delta\, F_{(1)}\left[\begin{matrix}a; & b, b'; x, y\\ c; & \end{matrix}\right] = x\,\frac{ab}{c}\, F_{(1)}\left[\begin{matrix}a+1; & b+1, b'; x, y\\ c+1; & \end{matrix}\right],$$

$$(\delta+\delta'+a)\, F_{(1)}\left[\begin{matrix}a; & b, b'; x, y\\ c; & \end{matrix}\right] = a\, F_{(1)}\left[\begin{matrix}a+1; & b, b'; x, y\\ c; & \end{matrix}\right],$$

$$(\delta+\delta'+c-1)\, F_{(1)}\left[\begin{matrix}a; & b, b'; x, y\\ c; & \end{matrix}\right] = (c-1)\, F_{(1)}\left[\begin{matrix}a; & b, b'; x, y\\ c-1; & \end{matrix}\right],$$

and from such relations we may deduce that $F_{(1)}$ is a solution of the partial differential equations

$$(1)\quad \left\{\begin{aligned} &[\delta(\delta+\delta'+c-1) - x(\delta+\delta'+a)(\delta+b)]\,z = 0,\\ &[\delta'(\delta+\delta'+c-1) - y(\delta+\delta'+a)(\delta'+b')]\,z = 0.\end{aligned}\right.$$

Operating on the first of these by δ' and the second by δ and subtracting we obtain a third equation which also has $F_{(1)}$ as a solution, namely

$$(2)\quad [x\delta'(\delta+b) - y\delta(\delta'+b')]\,z = 0.$$

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[1] 'Appell et Kampé de Fériet'. p.16.

This system of equations (1) and (2) has singularities corresponding to the lines $x = 0, 1, \infty$; $y = 0, 1, \infty$; $x = y$.[1]

Picard[2] has shown that $F_{(1)}$ may be expressed as a simple definite integral

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} b, b'; x, y\right] = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}du,$$

$$R(a) > 0, \; R(c-a) > 0.$$

The five changes of variable

$$u = (1-v), \quad u = \frac{v}{1-x+vx}, \quad u = \frac{v}{1-y+vy}, \quad u = \frac{1-v}{1-vx}, \quad u = \frac{1-v}{1-vy},$$

leave unaltered the form of the integral, and lead immediately to the following transformations:

$$(3)\left\{\begin{aligned} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} b, b'; x, y\right] &= (1-x)^{-b}(1-y)^{-b'} F_{(1)}\left[\begin{matrix}c-a; \\ c;\end{matrix} b, b'; \frac{-x}{1-x}, \frac{-y}{1-y}\right], \\ &= (1-x)^{-a} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} c-b-b', b'; \frac{-x}{1-x}, \frac{y-x}{1-x}\right], \\ &= (1-y)^{-a} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} b, c-b-b'; \frac{x-y}{1-y}, \frac{-y}{1-y}\right], \\ &= (1-x)^{c-a-b}(1-y)^{-b'} F_{(1)}\left[\begin{matrix}c-a; \\ c;\end{matrix} c-b-b', b'; x, \frac{x-y}{1-y}\right], \\ &= (1-x)^{-b}(1-y)^{c-a-b'} F_{(1)}\left[\begin{matrix}c-a; \\ c;\end{matrix} b, c-b-b'; \frac{y-x}{1-x}; y\right]. \end{aligned}\right.$$

Picard has also shown that the function

$$(4) \qquad z(x,y) = \int_g^h u^{b+b'-c}(u-1)^{c-a-1}(u-x)^{-b}(u-y)^{-b'}du, \text{[3]}$$

where g and h are any two of the five quantities $0, 1, \infty, x, y$, is an integral of the system, provided of course that the exponents

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[1] Erdélyi. 1. p. 141.

[2] vide 'Appell et Kampé de Fériet' p. 29 (4).

[3] ibid. p. 55.

$$1(c)\begin{cases} x^{1-c} F_{(2)}\left[1+a-c; \begin{matrix}1+b-c, b' \\ 2-c, c'\end{matrix}; x, y\right], \quad y^{1-c'} F_{(2)}\left[1+a-c'; \begin{matrix}b, 1+b'-c' \\ c, 2-c'\end{matrix}; x, y\right] \\ \qquad x^{1-c} y^{1-c'} F_{(2)}\left[2+a-c-c'; \begin{matrix}1+b-c, 1+b-c' \\ 2-c, 2-c'\end{matrix}; x, y\right]. \end{cases}$$

The Appell function

$$2(a) \qquad F_{(3)}\left[c; {}^{a, a', b, b'}; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{m!\, n!\, (c)_{m+n}} x^m y^n$$

is a solution of

$$2(b)\begin{cases} [\delta(\delta+\delta'+c-1) - x(\delta+a)(\delta+b)]z = 0, \\ [\delta'(\delta+\delta'+c-1) - y(\delta'+a')(\delta'+b')]z = 0. \end{cases}$$

The Horn series

$$3(a) \qquad G_{(2)}(a, a', b, b'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_{n-m} (b')_{m-n}}{m!\, n!} x^m y^n$$

is a solution of the pair of equations

$$3(b)\begin{cases} [\delta(\delta-\delta'-b) + x(\delta-\delta'+b')(\delta+a)]z = 0 \\ [\delta'(\delta'-\delta-b') + y(\delta'-\delta+b)(\delta'+a')]z = 0. \end{cases}$$

Finally, the Horn series

$$4(a) \qquad H_{(2)}(a; b, b', c'; d; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m-n} (b)_m (b')_n (c')_n}{(d)_m\, m!\, n!} x^m y^n$$

is a solution of

$$4(b)\begin{cases} [\delta(\delta+d-1) - x(\delta-\delta'+a)(\delta+b)]z = 0, \\ [\delta'(\delta'-\delta-a) + y(\delta'+b')(\delta'+c')]z = 0. \end{cases}$$

By setting $z = x^{1-d} z'$ we see that another solution of 4(b) is

$$x^{1-d} H_{(2)}(1+a-d; 1+b-d, b', c', 2-d; x, y).$$

4.3. THE SIXTY SOLUTIONS OF VAVASSEUR.

To find a solution corresponding to the singularities $x = \infty, y = \infty$ we change the variables to $X = \frac{1}{x}$, $Y = \frac{1}{y}$. We then have

$$x\frac{d}{dx} \equiv -X\frac{d}{dX} \quad ; \quad y\frac{d}{dy} \equiv -Y\frac{d}{dY}$$

so that we replace δ by $-\delta$, δ' by $-\delta'$, and equations 4.1.(1) transform to

$$\begin{cases} [(\delta-b)(\delta+\delta'-a) - X\delta(\delta+\delta'-c+1)]z = 0 \\ [(\delta'-b')(\delta+\delta'-a) - Y\delta'(\delta+\delta'-c+1)]z = 0 \end{cases}$$

Setting $z = X^b Y^{b'} z'$ we see that z' satisfies

$$\begin{cases} [\delta(\delta+\delta'-a+b+b') - X(\delta+b)(\delta+\delta'-c+1+b+b')]z' = 0, \\ [\delta'(\delta+\delta'-a+b+b') - Y(\delta'+b')(\delta+\delta'-c+1+b+b')]z' = 0, \end{cases}$$

which by comparison with 4.1.(1). have a solution of the type $F_{(1)}$, and the original equations have therefore the solution

(1) $$x^{-b}y^{-b'}F_{(1)}\left[\begin{matrix}1+b+b'-c; & b, b'; \frac{1}{x}, \frac{1}{y} \\ 1+b+b'-a; & \end{matrix}\right],$$

which is z_3 of the table of solutions in 'Appell and Kampé de Fériet'.

To find a solution corresponding to the singularities $x=1$, $y=1$, we set $X=(1-x)$, $Y=(1-y)$. We then have

$$x\frac{d}{dx} \equiv (1-\frac{1}{X})X\frac{d}{dX}, \text{ etc.}$$ so that we replace δ by $(1-\frac{1}{X})\delta$, δ' by $(1-\frac{1}{Y})\delta'$, and the first of the equations 4.1.(1). becomes.

$$[\delta\{(1-\tfrac{1}{X})\delta + (1-\tfrac{1}{Y})\delta' + c - 1\} + X\{(1-\tfrac{1}{X})\delta + (1-\tfrac{1}{Y})\delta' + a\}\{(1-\tfrac{1}{X})\delta + b\}]z = 0.$$

This simplifies to

(2) $$[\delta(\delta+a+b-c) + \frac{X}{Y}\delta'(\delta+b) - X(\delta+\delta'+a)(\delta+b)]z = 0,$$

and similarly the second equation 4.1.(1). becomes

(3) $$[\delta'(\delta'+a+b'-c) + \frac{Y}{X}\delta(\delta'+b') - Y(\delta+\delta'+a)(\delta'+b')]z = 0.$$

To simplify these equations further we make use of the auxiliary equation 4.1.(2). With the given change of variables this becomes

$$(1-x)(1-\tfrac{1}{Y})\,\delta'\{(1-\tfrac{1}{X})\delta+b\}z = (1-Y)(1-\tfrac{1}{X})\,\delta\,\{(1-\tfrac{1}{Y})\delta'+b'\}z,$$

which reduces to

(4) $$X\delta'(\delta+b)z = Y\delta(\delta'+b')z,$$

and it is interesting to notice that 4.1.(2) has reproduced itself with this change of variables. By means of equation (4) we can get rid of the terms in $\frac{X}{Y}$, $\frac{Y}{X}$, in (2) and (3) which become

(5) $$\begin{cases} [\delta(\delta+\delta'+a+b+b'-c) - X(\delta+\delta'+a)(\delta+b)]\,z = 0, \\ [\delta'(\delta+\delta'+a+b+b'-c) - Y(\delta+\delta'+a)(\delta'+b')]\,z = 0. \end{cases}$$

These equations are of the type $F_{(1)}$ and therefore the original equations have a solution

(6) $$F_{(1)}\left[\begin{matrix} a; \\ 1+a+b+b'-c; \end{matrix}\ b, b'\ ; 1-x, 1-y\right],$$

which is Z_2 of the table.

To deal with the singularity $x=y$, perhaps the most convenient transformation is $X = \frac{y}{x}$, $Y = y$.

Then $\frac{d}{dx} \equiv \frac{-y}{x^2}\frac{d}{dX}$ so that $x\frac{d}{dx} \equiv -X\frac{d}{dX}$,

and $\frac{d}{dy} \equiv \frac{1}{x}\frac{d}{dX} + \frac{d}{dY}$ so that $y\frac{d}{dy} \equiv X\frac{d}{dX} + Y\frac{d}{dY}$.

In 4.1.(1). we therefore replace δ by $-\delta$, and δ' by $\delta+\delta'$, and the equations become

$$[-\delta(\delta'+c-1) - \frac{Y}{X}(\delta'+a)(-\delta+b)]\,z = 0$$

$$[(\delta+\delta')(\delta'+c-1) - Y(\delta'+a)(\delta+\delta'+b')]\,z = 0$$

Setting $z = X^b Y^{1-c}\zeta$ we see that ζ satisfies the equations

(7) $$[X\delta'(\delta+b) - Y\,\delta(\delta'+1+a-c)]\,\zeta = 0$$

(8) $$[\delta'(\delta+\delta'+1+b-c) - Y(\delta+\delta'+1+b+b'-c)(\delta'+1+a-c)]\,\zeta = 0.$$

We operate on (7) with $(\delta+\delta'+b+b'-c)$ and on (8) with δ, subtract and drop a factor δ' to obtain

(9) $$[\delta(\delta+\delta'+1+b-c)-x(\delta+\delta'+1+b+b'-c)(\delta+b)]\zeta = 0.$$

Equations (8) and (9) are a pair of the type $F_{(1)}$, and therefore the original equations have a solution

(10) $$x^{-b}y^{1+b-c}F_{(1)}\left[\begin{matrix}1+b+b'-c; \\ 2+b-c;\end{matrix}\; b, 1+a-c\; ; \frac{y}{x}, y\right],$$

and by symmetry there is another solution

(11) $$x^{1+b'-c}y^{-b'}F_{(1)}\left[\begin{matrix}1+b+b'-c; \\ 2+b'-c;\end{matrix}\; 1+a-c, b'\; ; x, \frac{x}{y}\right],$$

which are respectively Z_{15}, and Z_{14} in the table of sixty.

Thus we see that the changes of variable (x,y), $(\frac{1}{x},\frac{1}{y})$, $(1-x,1-y)$, $(\frac{y}{x},y)$, $(x,\frac{x}{y})$ in each case transforms the original equations 4.1.(1). into a pair of equations of the same type. The pairs of equations may be most conveniently denoted by their solutions, and we thus have the transformations

$$I \equiv F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix}\; b, b'\; ; x, y\right],$$

$$A \equiv x^{-b}y^{-b'}F_{(1)}\left[\begin{matrix}1+b+b'-c; \\ 1+b+b'-a;\end{matrix}\; b, b'\; ; \frac{1}{x}, \frac{1}{y}\right],$$

$$B \equiv F_{(1)}\left[\begin{matrix}a; \\ 1+a+b+b'-c;\end{matrix}\; b, b'\; ; 1-x, 1-y\right],$$

$$C \equiv x^{-b}y^{1+b-c}F_{(1)}\left[\begin{matrix}1+b+b'-c; \\ 2+b-c;\end{matrix}\; b, 1+a-c\; ; \frac{y}{x}, y\right],$$

$$D \equiv x^{1+b'-c}y^{-b'}F_{(1)}\left[\begin{matrix}1+b+b'-c; \\ 2+b'-c;\end{matrix}\; 1+a-c, b'\; ; x, \frac{x}{y}\right].$$

These transformations of the equations A, B, C, D may be used to obtain further solutions. Thus for example

$$A.B. \equiv (1-x)^{-b}(1-y)^{-b'}F_{(1)}\left[\begin{matrix}c-a; \\ 1+b+b'-a;\end{matrix}\; b, b'\; ; \frac{1}{1-x}, \frac{1}{1-y}\right],$$

and $$B.A \equiv x^{-b}y^{-b'}F_{(1)}\left[\begin{matrix}1+b+b'-c \\ 1+a+b+b'-a;\end{matrix}\; b, b'; \frac{x-1}{x}, \frac{y-1}{y}\right],$$

which are Z_{13} and Z_{12} respectively.

The transformations I, A, B, generate the harmonic group with six members in which the variables are

(x,y), $(\frac{1}{x},\frac{1}{y})$, $(1-x,1-y)$, $(\frac{1}{1-x},\frac{1}{1-y})$, $(\frac{x-1}{x},\frac{y-1}{y})$, $(\frac{x}{x-1},\frac{y}{y-1})$,

and together with the further transformations C and D generate a group of 120 pairs of differential equations and 120 corresponding solutions of the type $F_{(1)}$. These solutions reduce to the sixty solutions of Vavasseur since each is duplicated by the variable change $(x,y)\to(y,x)$, which is obtained for example by the variable change equivalent to CADC.

4.4. SOLUTIONS IN TERMS OF SERIES OF THE TYPES $F_{(2)}$, $F_{(3)}$, $G_{(2)}$, $H_{(2)}$.

In the harmonic transformations I. A. B. we have treated x and y similarly. If we change only one of the variables, considering for example $(X, Y) = (x, 1-y), (x, \frac{1}{y}), (x, \frac{1}{1-y})$, etc. then the resulting equations are no longer of the type $F_{(1)}$ and we obtain solutions in terms of other hypergeometric series. Each series solution is typical of sixty since we may transform any of the sixty pairs of equations obtained in the last section.

With the change of variable $X=x$, $Y=(1-y)$, $\delta\to\delta$, $\delta'\to(1-\frac{1}{Y})\delta'$, equations 4.1(1). become

(1) $[\delta(\delta+\delta'+c-1)-\frac{1}{Y}\delta\delta'-X(\delta+\delta'+a)(\delta+b)+\frac{X}{Y}\delta'(\delta+b)]z=0,$

(2) $[\delta'(\delta'+a+b'-c)-Y(\delta+\delta'+a)(\delta'+b')]z=0,$

and the auxiliary equation 4.1.(2). becomes

(3) $$[X\delta'(\delta+b) + Y\delta(\delta'+b') - \delta\delta']z = 0.$$

This enables us to replace the terms $[\frac{X}{Y}\delta'(\delta+b) - \frac{1}{Y}\delta\delta']z$ by $[-\delta(\delta'+b')]z$ in (1) to obtain

(4) $$[\delta(\delta+c-b'-1) - X(\delta+\delta'+a)(\delta+b)]z = 0.$$

Reference to 4.2. shows that equations (2) and (4) are of the type $F_{(2)}$, and thus the original equations have the solutions

(5) $$\left\{\begin{array}{l} F_{(2)}\left[\begin{matrix}a; & b, & b' \\ & c-b', & 1+a+b'-c\end{matrix}; x, 1-y\right], \\ x^{1+b'-c}F_{(2)}\left[\begin{matrix}1+a+b'-c; & 1+b+b'-c, & b' \\ & 2+b'-c, & 1+a+b'-c\end{matrix}; x, 1-y\right], \\ (1-y)^{c-a-b'}F_{(2)}\left[\begin{matrix}c-b'; & b, & c-a \\ & c-b', & 1+c-a-b'\end{matrix}; x, 1-y\right], \\ x^{1+b'-c}(1-y)^{c-a-b'}F_{(2)}\left[\begin{matrix}1; & 1+b+b'-c, & c-a \\ & 2+b'-c, & 1+c-a-b'\end{matrix}; x, 1-y\right]. \end{array}\right.$$ (1)

These $F_{(2)}$ series are of course all specialised, there being only four parameters to occupy the five parameter positions.

To deal with the region $(0,\infty)$ we set $X = x$, $Y = \frac{1}{y}$ and replace δ' by $-\delta'$. The equations become

$$[\delta(\delta-\delta'+c-1) - X(\delta-\delta'+a)(\delta+b)]z = 0,$$
$$[(\delta'-b')(\delta'-\delta-a) - Y\delta'(\delta'-\delta-c+1)]z = 0,$$

or setting $z = Y^{b'}\zeta$ we see that ζ satisfies

(6) $$\left\{\begin{array}{l} [\delta(\delta-\delta'+c-b'-1) - X(\delta-\delta'+a-b')(\delta+b)]\zeta = 0, \\ [\delta'(\delta'-\delta-a+b') - Y(\delta'-\delta-c+1+b')(\delta'+b')]\zeta = 0. \end{array}\right.$$

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(1) Erdélyi. 1. p. 151.

Reference to 4.2. shows that equations (6) are of the type $G_{(2)}$ and that the original equations have a solution

(7) $$y^{-b'} G_{(2)}(b, b', 1+b'-c, a-b'; -x, -\tfrac{1}{y}),^{(1)}$$

and as usual this is typical of sixty solutions. For example, just as we have obtained this solution by transforming the pair of equations determined by the solution $F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix}\ b, b'; x, y\right]$, we may by performing a similar transformation on the equations denoted by the solution $F_{(1)}\left[\begin{matrix}a; \\ 1+a+b+b'-c;\end{matrix}\ b, b'; 1-x, 1-y\right]$ obtain the $G_{(2)}$ series

$$(1-y)^{-b'} G_{(2)}(b, b', c-a-b', a-b'; x-1, \tfrac{1}{y-1}).$$

It is not of course necessary to work with the equations but merely to make the parameter changes equivalent to

$$F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix}\ b, b'; x, y\right] \longrightarrow y^{-b'} G_{(2)}(b, b', 1+b'-c, a-b'; -x, -\tfrac{1}{y}).$$

To obtain a solution in the variables $x, \frac{1}{1-y}$ we set $Y' = \frac{1}{Y}$ in equations (3) and (4) of this section, replace δ' by $-\delta'$ and obtain

$$[(\delta'-b')(\delta'-\delta-a) - Y'\delta'(\delta'-a-b'+c)]z = 0,$$

$$[\delta(\delta+c-b'-1) - X(\delta-\delta'+a)(\delta+b)]z = 0.$$

Set $z = Y'^{b'}\zeta$ and we have

(8) $$\begin{cases} [\delta'(\delta'-\delta-a+b') - Y'(\delta'+b')(\delta'-a+c)]\zeta = 0, \\ [\delta(\delta+c-b'-1) - X(\delta-\delta'+a-b')(\delta+b)]\zeta = 0, \end{cases}$$

which are of the type $H_{(2)}$ and reference to 4.2. shows that the original equations have the solutions

(1) Erdélyi. 1. p. 133.(6).

$$(9)\begin{cases} (1-y)^{-b'} H_{(2)}\left(a-b';\, b, b', c-a;\, c-b';\, x, \frac{1}{y-1}\right), \\ (1-y)^{-b'} x^{1+b'-c} H_{(2)}\left(1+a-c;\, 1+b+b'-c, b', c-a;\, 2+b'-c;\, x, \frac{1}{y-1}\right). \end{cases}$$

Setting $Y' = (1-Y)$ in equations (6) and the corresponding auxiliary equation which is

$$[\, X\, Y(\delta+b)(\delta'+b') - \delta\delta' \,]\, \zeta = 0,$$

we obtain after some simple manipulation

$$(10)\begin{cases} [\,\delta'(\delta'+a+b'-c) - Y'(\delta'-\delta-c+1+b')(\delta'+b')\,]\,\zeta = 0 \\ [\,\delta\,(\delta-\delta'+c-b'-1) - X\,(\delta+a)(\delta+b)\,]\,\zeta = 0 \end{cases}$$

showing that the original equations have solutions

$$(11)\begin{cases} y^{-b'} H_{(2)}\left(1+b'-c;\, b', a, b;\, 1+a+b'-c;\, \frac{y-1}{y}, -x\right), \\ (y-1)^{c-a-b'} y^{a-c} H_{(2)}\left(1-a;\, c-a, a, b;\, 1+c-a-b';\, \frac{y-1}{y}, -x\right). \end{cases}$$

Finally setting $Y'' = \frac{1}{Y'}$ in equations (10) above, and then $\zeta' = Y''^{\,b'}\zeta$ we obtain

$$(12)\begin{cases} [\,\delta'(\delta+\delta'+c-1) - Y''(\delta'+b')(\delta'+c-a)\,]\,\zeta' = 0, \\ [\,\delta(\delta+\delta'+c-1) - X(\delta+a)(\delta+b)\,]\,\zeta' = 0, \end{cases}$$

where X and Y'' in terms of the original variables are now equal to x and $\frac{y}{y-1}$. Reference to 4.2. shows that these equations are of the type $F_{(3)}$ and we are led to the solution

$$(13)\qquad (1-y)^{-b'} F_{(3)}\left[c;\, {}^{a, b', b, c-a}\,;\, x, \frac{y}{y-1}\right].$$

This completes all the possible combinations of variables arising from the transformations $x \to \frac{1}{x}, 1-x, \frac{y}{x}$, etc. together with the corresponding transformations for y.

4.5. RELATIONS BETWEEN THE SERIES.

All these solutions arise as various ways of expressing Erdélyi's total of twenty five, ten of the type $F_{(1)}$, and fifteen of the type $G_{(2)}$. In addition to the transformations 4.1.(3) by which any $F_{(1)}$ can be expressed in five different ways as another series of the type $F_{(1)}$ we also have

$$(1)\begin{cases} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} b,b'; x, y\right] = (1-y)^{-b'} F_{(3)}\left[c; ^{a,c-a,b,b'}; x, \frac{y}{y-1}\right], \\ \qquad = (1-x)^{-b} F_{(3)}\left[c; ^{c-a,a,b,b'}; \frac{x}{x-1}, y\right], \end{cases}$$ [1]

$$(2)\begin{cases} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} b,b'; x,y\right] = \left(\frac{x}{y}\right)^{b'} F_{(2)}\left[b+b'; \begin{matrix}a, & b' \\ c, & b+b'\end{matrix}; x, 1-\frac{x}{y}\right], \\ \qquad = \left(\frac{y}{x}\right)^{b} F_{(2)}\left[b+b'; \begin{matrix}a, & b \\ c, & b+b'\end{matrix}; 1-\frac{y}{x}, y\right], \end{cases}$$ [2]

$$(3)\begin{cases} F_{(1)}\left[\begin{matrix}a; \\ c;\end{matrix} b,b'; x,y\right] = \left(1-\frac{y}{x}\right)^{-b'} H_{(2)}\left(b; a, b', 1-b; c; x, \frac{y}{x-y}\right), \\ \qquad = \left(1-\frac{x}{y}\right)^{-b} H_{(2)}\left(b'; a, b, 1-b'; c; \frac{x}{y-x}, y\right). \end{cases}$$ [3]

Erdélyi[4] also obtains an integral representation of a $G_{(2)}$ series from which he is able to deduce the following transformations

$$G_{(2)}(a,a',b,b'; x,y) = (1+x)^{-b'}(1-xy)^{-a'} G_{(2)}\left(1-a-b, a', b, b'; \frac{-x}{x+1}, \frac{y+xy}{1-xy}\right),$$

$$= (1+y)^{-b}(1-xy)^{-a} G_{(2)}\left(a, 1-a'-b', b, b'; \frac{x+xy}{1-xy}, \frac{-y}{y+1}\right),$$

$$= (1+x)^{-b'}(1+y)^{-b}(1-xy)^{1-a-a'} .$$

$$. G_{(2)}\left(1-a-b, 1-a'-b', b, b', \frac{-x-xy}{1+x}, \frac{-y-xy}{1+y}\right),$$

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(1) 'Appell et Kampé de Fériet' p. 24. eq'ns (29)(29'). (2) ibid. p. 35. eq'n (9)

(3), (4) Erdélyi. I. p. 146. eq'n (13) ; page 148, eq'ns (17) – (20).

and also

(4) $$G_{(2)}(a,a',b,b';x,y) = (1+x)^{-a}(1+y)^{-a'} F_{(2)}\left[{1-b-b';a,a' \atop 1-b,1-b'};\frac{x}{x+1},\frac{y}{y+1}\right],$$

(5) $$G_{(2)}(a,a',b,b';x,y) = (1+y)^{-a'} H_{(2)}(b';a,a',1-b-b',1-b;-x,\frac{-y}{y+1}).$$ [1]

The continuation of any of the series into a different domain of convergence is perhaps best tackled by considering Erdélyi's integral representations. Nevertheless it is possible to obtain relations between more than two solutions by making use of Gauss' formulae for the continuation of the ordinary hypergeometric series, and we conclude with an example of this method.

$$F_{(1)}\left[{a; \atop c;} b,b';x,y\right] = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{m!(c)_m}x^m {}_2F_1\left[{a+m,b' \atop c+m};y\right],$$

$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{m!(c)_m}x^m\left\{\begin{aligned}&(-)^{a+m}\frac{\Gamma(c+m)\Gamma(b'-a-m)}{\Gamma(c-a)\Gamma(b')}y^{-a-m}{}_2F_1\left[{a+m,\ 1+a-c \atop 1+a-b'+m};\frac{1}{y}\right]\\&+(-)^{b'}\frac{\Gamma(c+m)\Gamma(a-b'+m)}{\Gamma(a+m)\Gamma(c-b'+m)}y^{-b'}{}_2F_1\left[{b',\ 1+b'-c-m \atop 1+b'-a-m};\frac{1}{y}\right]\end{aligned}\right\},$$ [2]

$$=\frac{\Gamma(c)\Gamma(b'-a)}{\Gamma(c-a)\Gamma(b')}(-)^a y^{-a}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(1+a-c)_n}{m!\,n!\,(1+a-b')_{m+n}}\left(\frac{x}{y}\right)^m y^{-n}$$

$$+\frac{\Gamma(c)\Gamma(a-b')}{\Gamma(a)\Gamma(c-b')}(-)^b y^{-b}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a-b')_m(b)_m(b')_n(1+b'-c-m)_n}{m!\,n!\,(c-b')_m(1+b'-a-m)_n}x^m y^{-n},$$

and therefore

$$F_{(1)}\left[{a; \atop c;} b,b';x,y\right] = \frac{\Gamma(c)\Gamma(b'-a)}{\Gamma(c-a)\Gamma(b')}(-)^a y^{-a} F_{(1)}\left[{a; \atop 1+a-b';} b,1+a-c;\frac{x}{y},\frac{1}{y}\right]$$
$$+\frac{\Gamma(c)\Gamma(a-b')}{\Gamma(a)\Gamma(c-b')}(-)^b y^{-b} G_{(2)}(b,b';a-b,1+b'-c;-x,-\frac{1}{y}).$$

By means of the equations (1) - (5) we may express any of the solutions of 4.3. and 4.4. as one of the ten essentially

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(1) Erdélyi. 1. p. 150. (22).

(2) 'Appell et Kampé de Fériet'. p. 7. (10').

different $F(1)$ solutions or the fifteen essentially different $G(2)$ solutions, and by the method outlined above we may obtain relations between three or more of the solutions.

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CHAPTER V

THE DIFFERENTIAL EQUATION SATISFIED BY THE GENERAL ${}_3F_2$.

5.1. INTRODUCTORY REMARKS. The differential equation

(1) $$[\delta(\delta+b_1-1)(\delta+b_2-1) - x(\delta+a_1)(\delta+a_2)(\delta+a_3)]\,y = 0$$

with singularities at $x = 0, 1, \infty$, has solutions valid at the origin

$$ {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; x\right],$$

$$x^{1-b_1}\,{}_3F_2\left[\begin{matrix} 1+a_1-b_1, 1+a_2-b_1, 1+a_3-b_1 \\ 2-b_1, 1+b_2-b_1 \end{matrix}; x\right],\quad x^{1-b_2}\,{}_3F_2\left[\begin{matrix} 1+a_1-b_2, 1+a_2-b_2, 1+a_3-b_2 \\ 2-b_2, 1+b_1-b_2 \end{matrix}; x\right].$$

Setting $x' = \frac{1}{x}$ so that $\delta' \equiv -\delta$, equation (1) becomes

(2) $$[(\delta'-a_1)(\delta'-a_2)(\delta'-a_3) - x'\,\delta'(\delta'+1-b_1)(\delta'+1-b_2)]\,y = 0$$

and there are solutions relating to the singularity at infinity of which the first is

(3) $$x^{-a_1}\,{}_3F_2\left[\begin{matrix} a_1, 1+a_1-b_1, 1+a_1-b_2 \\ 1+a_1-a_2, 1+a_1-a_3 \end{matrix}; \frac{1}{x}\right],$$

and the other two are obtained by permuting a_1, a_2, a_3.

In this chapter we obtain solutions of (1) relating to the singularity $x = 1$, i.e. solutions in ascending and descending powers of $1-x$. This problem has already been considered by Darling.[1] Here, using the δ-notation we are able to simplify Darling's technique and extend somewhat his results.

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[1] Darling .1.

5.2. BURCHNALL'S LEMMA. To transform the equation, changing the variable to $1-x$, we make use of the following lemma due to Burchnall.

Lemma. If $f(\delta) = (\delta+a_1)(\delta+a_2)\cdots(\delta+a_n)$, and $D = \frac{d}{dx}$

$$\text{then } f(\delta - D) = f(\delta) - \Delta f(\delta).D + \frac{\Delta^2}{2!} f(\delta).D^2 - \cdots \pm D^n;$$

$$= f(\delta) - x^{-1}\delta\,\Delta f(\delta-1) + \frac{x^{-2}}{2!}\,\delta(\delta-1)\,\Delta^2 f(\delta-2) - \cdots \pm x^{-n}\,\delta(\delta-1)\cdots(\delta-n+1),$$

$$\text{where } \Delta f(\delta) = f(\delta+1) - f(\delta), \quad \Delta^r f(\delta) = \Delta.\Delta^{r-1} f(\delta)$$
$$= \sum_{s=0}^{r} (-)^s \binom{r}{s} f(\delta+r-s).$$

The lemma is easily proved for $n=1$ since

$$(\delta - D + a_1) = (\delta + a_1) - x^{-1}\delta.$$

Assume that it is true for n and let

$$g(\delta) = (\delta + a_{n+1}) f(\delta); \qquad \text{then since}$$

$$D\,\phi(\delta) = x^{-1}\delta\,\phi(\delta) = \phi(\delta+1)x^{-1}\delta = \phi(\delta+1)D, \quad \text{we have}$$

$$(\delta + a_{n+1} - D)\, f(\delta - D) = (\delta + a_{n+1}) f(\delta - D) - f(\delta + 1 - D).D.$$

In this the coefficient of $\frac{\pm D^r}{r!}$ is

$$(\delta + a_{n+1})\,\Delta^r f(\delta) + r\,\Delta^{r-1} f(\delta+1)$$

$$= (\delta + a_{n+1})\,\Delta^r f(\delta) + r\,\Delta^{r-1}(\Delta+1) f(\delta)$$

$$= (\delta + a_{n+1} + r)\,\Delta^r f(\delta) + r\,\Delta^{r-1} f(\delta)$$

$$= \sum_{r=0}^{s} (\delta + a_{n+1} + r - s + s)(-)^s \binom{r}{s} f(\delta + r - s) + \quad r\,\Delta^{r-1} f(\delta)$$

$$= \sum_{r=0}^{s} (-)^s \binom{r}{s} g(\delta + r - s) + r \sum_{s=1}^{r} (-)^s \binom{r-1}{s-1} f(\delta + r - s) \; + r\,\Delta^{r-1} f(\delta)$$

$$= \Delta^r g(\delta) - r\sum_{s=0}^{r-1}(-)^s\binom{r-1}{s} f(\delta+r-1-s) + r\,\Delta^{r-1} f(\delta),$$

$$= \Delta^r g(\delta),$$

and the lemma is established by induction.

5.3. SOLUTIONS OF THE EQUATION IN ASCENDING POWERS OF $(1-x)$.

Equation 5.1. (1) may be written

$$(\delta+b_1)(\delta+b_2)\,D\,y = (\delta+a_1)(\delta+a_2)(\delta+a_3)\,y, \quad \text{or}$$

(1) $$g(\delta)\,Dy = f(\delta)\,y,$$

where

(2) $$g(\delta) = (\delta+b_1)(\delta+b_2) \quad ; \quad f(\delta) = (\delta+a_1)(\delta+a_2)(\delta+a_3).$$

If we set $\xi = (1-x)$ we must replace δ by $(\delta-D)$ and (i) becomes

$$g(\delta-D)Dy + f(\delta-D)\,y = 0,$$

or using Burchnall's lemma

$$[g(\delta)\,D - \Delta g(\delta).D^2 + D^3 + f(\delta) - \Delta f(\delta).D + \tfrac{\Delta^2}{2!} f(\delta).D^2 - D^3]\,y = 0,$$

i.e.

$$[\,f(\delta) - \xi^{-1}\delta\{\Delta f(\delta-1) - g(\delta-1)\} + \xi^{-2}\delta(\delta-1)\{\tfrac{\Delta^2}{2!} f(\delta-2) - \Delta g(\delta-2)\}]\,y = 0,$$

or multiplying by ξ^2 we obtain finally

(3) $$[\,\delta(\delta-1)\{\tfrac{\Delta^2}{2!} f(\delta-2) - \Delta g(\delta-2)\} - \xi\delta\{\Delta f(\delta-1) - g(\delta-1)\} + \xi^2 f(\delta)]\,y = 0.$$

Since $\frac{\Delta^2}{2!} f(\delta-2) - \Delta g(\delta-2) = (\delta + a_1 + a_2 + a_3 - b_1 - b_2)$,

this equation therefore has solutions led by a constant, by ξ, and by ξ^s where $s = b_1+b_2-a_1-a_2-a_3$, or alternatively the equation has two independent solutions led by a constant, together with one led by ξ^s. We assume that s does not take the value 0 or 1 so that the indicial equation does not have equal roots.

? s integral

Equation (3) is an example of a 'three-term' differential equation, that is to say it is of the type

(4) $$[f_0(\delta) + x f_1(\delta) + x^2 f_2(\delta)]\, y = 0.$$

If in such an equation we assume a trial solution

$$y = \sum_{n=0}^{\infty} a_n x^{\alpha+n} \quad , \quad \text{where } f_0(\alpha) = 0,$$

then the coefficients a_n must satisfy the relation

(5) $$a_n f_0(\alpha+n) + a_{n-1} f_1(\alpha+n-1) + a_{n-2} f_2(\alpha+n-2) = 0,$$

and in particular this relation must still be satisfied when $n=1$, so that

(6) $$a_1 f_0(\alpha+1) + a_0 f_1(\alpha) = 0.$$

For (6) to be satisfied there are two possibilities. Either

(7) $$\begin{cases} \text{(i)} & f_0(\alpha+1) = 0; \quad f_1(\alpha) = 0 \quad , \quad \text{or} \\ \text{(ii)} & \dfrac{a_0}{a_1} = \dfrac{-f_0(\alpha+1)}{f_1(\alpha)} . \end{cases}$$

The first of these conditions implies that the original equation has a solution beginning with $x^{\alpha+1}$, and that the equation may be written

$$[(\delta-\alpha)(\delta-\alpha-1) f_0'(\delta) + x(\delta-\alpha) f_1'(\delta) + x^2 f_2(\delta)]\, y = 0.$$

Equation (3) is of this type when $\alpha = 0$, so that provided we can find some a_n satisfying (5) for $n \geq 2$, then $\sum_1 a_n \xi^n$ is a solution. If however we set $y = \xi^s z$ in (3) in order to find the solution led by ξ^s then the condition (7)(i) no longer holds, and in this case it is not sufficient to find coefficients a_n satisfying (5), it is necessary in addition that a_0, a_1 should satisfy (7)(ii).[1]

The problem of finding a solution of (3) led by a constant term is equivalent therefore to finding coefficients a_n satisfying the three-term recurrence formula

$$(8)\quad (n+2)(n+1)\left\{\frac{\Delta^2}{2}f(n) - \Delta g(n)\right\}a_{n+2} + (n+1)\{\Delta f(n) - g(n)\}a_{n+1} + f(n)a_n = 0,$$

$$\text{for } n \geqslant 2.$$

Using the methods of the last chapter we consider

$$F_{n+2} = {}_3F_2\left[\begin{matrix} A_1, A_2, A_3 \\ 1+B_1, 1+B_2+n+2 \end{matrix};\right],$$

which is annihilated by the operator

$$[\delta(\delta+B_1)(\delta+B_2+n+2) - (\delta+A_1)(\delta+A_2)(\delta+A_3)].$$

This operator may be written

$$[P + Q(\delta+B_2+n+2) + R(\delta+B_2+n+2)(\delta+B_2+n+1)]$$

and setting $\delta = -(B_2+n+2),\ -(B_2+n+1),\ -(B_2+n)$ in turn to evaluate P, Q, and R gives

$$P = G_1(n+1),$$

$$Q = -\{\Delta G_1(n) - G_2(n)\}$$

$$R = \left\{\frac{\Delta^2}{2}G_1(n-1) - \Delta G_2(n-1)\right\}.$$

where $G_1(n) = (n+1+B_2-A_1)(n+1+B_2-A_2)(n+1+B_2-A_3)$,

and $G_2(n) = (n+1+B_2-B_1)(n+1+B_2)$.

---

[1] This explains the anomoly mentioned by Darling. (Darling. 1. p.67.). In point of fact Darlings series (9) is not a solution of the equation since it fails to satisfy either (7)(i) or (7)(ii).

Thus F_n satisfies the recurrence relation

$$G_1(n+1)F_{n+2} + (n+\beta_2+2)\{\Delta G_1(n) - G_2(n)\}F_{n+1}$$
$$+ (n+\beta_2+2)(n+\beta_2+1)\{\tfrac{\Delta^2}{2}G_1(n-1) - \Delta G_2(n-1)\}F_n = 0.$$

We now make use of 2.14. to exchange the end coefficients and find that

$$\phi_n = \frac{(1+\beta_2-A_1)_n(1+\beta_2-A_2)_n(1+\beta_2-A_3)_n}{n!(1+\beta_2)_n(1+\beta_1+\beta_2-A_1-A_2-A_3)_n}F_n$$

satisfies the recurrence relation

(9) $$(n+2)(n+1)\{\tfrac{\Delta^2}{2}G(n) - \Delta G_2(n)\}\phi_{n+2}$$
$$-(n+1)\{\Delta G_1(n) - G_2(n)\}\phi_{n+1} + G_1(n)\phi_n = 0,$$

which is identical with (8) if

$$G_1(n) = f(n) \quad ; \quad G_2(n) = g(n).$$

$G_{(2)}(n)$ and $g(n)$ may be identified in two ways and we are led to the solutions of (3)

(10) $$\begin{cases} y = \displaystyle\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{n!(b_1)_n(1-s)_n}\,{}_3F_2\left[\begin{matrix}b_1-a_1, b_1-a_2, b_1-a_3\\ 1+b_1-b_2, b_1+n\end{matrix};\right]\xi^n, \\ y = \displaystyle\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{n!(b_2)_n(1-s)_n}\,{}_3F_2\left[\begin{matrix}b_2-a_1, b_2-a_2, b_2-a_3\\ 1+b_2-b_1, b_2+n\end{matrix};\right]\xi^n, \end{cases}$$ [1]

i.e. solutions of 5.1.(1) in ascending powers of $(1-x)$.

By transforming the ${}_3F_2$ in (10) using Thomae relations, various other solutions of the equation can be found, convergent under different conditions and in fact we may obtain the solutions (10)

---

(1) These solutions which are (4) and (5) of Darling.I. are shown by Darling to be independent.

by a formal process directly from ${}_3F_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; 1-\xi\right]$ which may be written

$$\sum_{r=0}^{\infty} \frac{(a_1)_r(a_2)_r(a_3)_r}{r!\,(b_1)_r(b_2)_r} \sum_{n=0}^{r} \frac{(-r)_n}{n!} \xi^n$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{n!\,(b_1)_n(b_2)_n}\, {}_3F_2\left[\begin{matrix}a_1+n, a_2+n, a_3+n\\ b_1+n,\ b_2+n\end{matrix};\right](-)^n \xi^n .$$

This expression obviously diverges, but it satisfies the difference relation (8) and may be transformed into the solutions (10) using the Thomae transformation[1]

$${}_3F_2\left[\begin{matrix}a_1+n, a_2+n, a_3+n\\ b_1+n,\ b_2+n\end{matrix};\right] = \frac{\Gamma(s-n)\Gamma(b_2-b_1)\Gamma(b_2+n)}{\Gamma(b_2-a_1)\Gamma(b_2-a_2)\Gamma(b_2-a_3)}\, {}_3F_2\left[\begin{matrix}b_1-a_1, b_1-a_2, b_1-a_3\\ 1+b_1-b_2,\ b_1+n\end{matrix};\right]$$

$$+ \text{idem } (b_1 \rightleftarrows b_2).$$

To find the solution of (3) led by ξ^s we set $y = \xi^s z$ when the equation to be satisfied by z is

$$(11)\quad [\delta(\delta+s)(\delta+s-1)z - \xi(\delta+s)\{\Delta f(\delta+s-1) - g(\delta+s)\} + \xi^2 f(\delta+s)]\, z = 0.$$

If we assume a series solution of the form $z = \sum A_n \xi^n$, $A_0 = 1$, then the coefficients A_n must satisfy

$$(12)\quad n(n+s)(n+s-1)A_n - (n+s-1)\{\Delta f(n+s-2) - g(n+s-2)\}A_{n-1} + f(n+s-1)A_{n-2} = 0,$$

and in particular

$$s(1+s)A_1 = s\{(s+a_1)(s+a_2)(s+a_3) - (s+a_1-1)(s+a_2-1)(s+a_3-1)$$
$$- (s+b_1-1)(s+b_2-1)\}$$
$$= s\{s(b_1+b_2) + a_1a_2 + a_2a_3 + a_3a_1 - b_1b_2\}$$
$$= s\{(b_1+b_2-a_1-a_3)(b_1+b_2-a_2-a_3) - (b_1-a_3)(b_2-a_3)\}.$$

That is to say $$A_1 = \frac{(b_1+b_2-a_1-a_3)(b_1+b_2-a_2-a_3)}{(1+s)}\, {}_3F_2\left[\begin{matrix}b_1-a_3, b_2-a_3, -1\\ b_1+b_2-a_1-a_3,\ b_1+b_2-a_2-a_3\end{matrix};\right].$$

---

[1] vide tract pps 18, 19; the relation connecting $F_p(4,0,5)$ with $F_n(5) + F_n(4)$.

We consider therefore $F_n = {}_3F_2\left[\begin{matrix} b_1-a_3, b_2-a_3, -n \\ b_1+b_2-a_1-a_3, b_1+b_2-a_2-a_3 \end{matrix}; \right]$ satisfying

$$[\delta(\delta+b_1+b_2-a_1-a_3-1)(\delta+b_1+b_2-a_2-a_3-1)-(\delta+b_1-a_3)(\delta+b_2-a_3)(\delta-n)]F_n = 0,$$

$$\text{i.e.} \quad [f_1(\delta)-g_1(\delta)]F_n = 0.$$

We now rearrange the operator

(13) $$f_1(\delta)-g_1(\delta) \equiv P+(\delta-n)Q+(\delta-n)(\delta-n+1)R,$$

and setting $\delta = n, n-1, n-2$ in turn we find values for P, Q, R.

(14) $P = f_1(n)$; $Q = f_1(n)-f_1(n-1)+g_1(n-1)$; $R = (b_1+b_2-a_1-a_2-2)$.

Performing the operations indicated by the rearranged operator we have

(15) $$P.F_n - nQ.F_{n-1} + n(n-1)R.F_{n-2} = 0.$$

From (14)

$$Q = n(n+s+a_1-1)(n+s+a_2-1)-(n-1)(n+s+a_1-2)(n+s+a_2-2)-(n+b_1-a_3-1)(n+b_2-a_3-1)$$

and therefore

$$\begin{aligned} Q &- \{\Delta f(n+s-2) - g(n+s-2)\} \\ &= -(s+a_3-1)(n+s+a_1-1)(n+s+a_2-1)+(s+a_3-1)(n+s+a_1-2)(n+s+a_2-2) \\ &\qquad +(n+s+b_1-2)(n+s+b_2-2), \\ &= -(s+a_3-1)(2n+2s+a_1+a_2-3)+(s-2)(2n+b_1+b_2) \\ &\qquad +(s+2)^2+(a_3+1)(2n+b_1+b_2)-(a_3+1)^2, \\ &= -(s+a_3-1)(2n+2s+a_1+a_2-3-2n-b_1-b_2-s+a_3+3), \\ &= 0. \end{aligned}$$

Thus the coefficients of $(n+s-1)A_{n-1}$ in (12) and nF_{n-1} in (15) are identical. To make the necessary exchanges of end coefficients in (15) we set $\phi_n = \dfrac{(b_1+b_2-a_2-a_3)_n(b_1+b_2-a_1-a_3)_n}{n!\,(1+s)_n}F_n$ and find that

Φ_n satisfies (12), so that we have as a solution of (3)

(16) $$y = \xi^s \sum_{n=0}^{\infty} \frac{(b_1+b_2-a_1-a_3)_n (b_1+b_2-a_2-a_3)_n}{n!\,(1+s)_n} \, {}_3F_2\left[\begin{matrix} b_1-a_3,\ b_2-a_3,\ -n \\ b_1+b_2-a_1-a_3,\ b_1+b_2-a_2-a_3 \end{matrix};\right] \xi^n .$$

Since the original equation is symmetrical with respect to a_1, a_2, a_3, and since there is only one solution led by ξ^s this means that

$$\frac{(b_1+b_2-a_1-a_3)_n (b_1+b_2-a_2-a_3)_n}{(1+s)_n} \, {}_3F_2\left[\begin{matrix} b_1-a_3,\ b_2-a_3,\ -n \\ b_1+b_2-a_1-a_3,\ b_1+b_2-a_2-a_3 \end{matrix};\right]$$

is invariant for permutations of a_1, a_2, a_3. Such permutations are in fact applications of the fundamental Thomae relations.[1]

5.4. SOLUTIONS IN DESCENDING POWERS OF $(1-x)$.

To obtain solutions in descending powers of $(1-x)$ we set $\zeta = \frac{1}{\xi}$ in equation (3) of the last section which then becomes

(1) $$\left[f(-\delta) + \zeta\,\delta\{\Delta f(-\delta-1) - g(-\delta-1)\} + \zeta^2\delta(\delta+1)\left\{\frac{\Delta^2}{2!} f(-\delta-2) - \Delta g(-\delta-2)\right\}\right] y = 0 .$$

The indicial equation $f(-\delta) = 0$ has roots a_1, a_2, a_3, so that the equation obtained by setting $y = \zeta^{a_1} z$, i.e.

(2) $$\left[f(-\delta-a_1) + \zeta(\delta+a_1)\{\Delta f(-\delta-1-a_1) - g(-\delta-1-a_1)\}\right.$$
$$\left. + \zeta^2(\delta+a_1)(\delta+a_1+1)\left\{\frac{\Delta^2}{2} f(-\delta-a_1-2) - \Delta g(-\delta-a_1-2)\right\}\right] z = 0$$

has a solution led by a constant term.

Assuming a solution $\sum_{n=0}^{\infty} A_n \zeta^n$, $A_0 = 1$, then A_n must satisfy the difference relation

(3) $$f(-n-a_1) A_n + (n+a_1-1)\{\Delta f(-n-a_1) - g(-n-a_1)\} A_{n-1}$$
$$+ (n+a_1-1)(n+a_1-2)\left\{\frac{\Delta^2}{2} f(-n-a_1) - \Delta g(-n-a_1)\right\} A_{n-2} = 0 .$$

---

(1) Darling has obtained the relation between the three solutions (10) and (16) and the ${}_3F_2\left[\begin{matrix} a_1,\ a_2,\ a_3 \\ b_1,\ b_2 \end{matrix}; 1-\xi\right]$. Darling. I. p. 68. (15).

In particular this must be satisfied when $n=1$, when

$f(-n-a_1+1)=0$ and

$$A_1 = a_1\left\{1-\frac{(1+a_1-b_1)(1+a_1-b_2)}{(1+a_1-a_2)(1+a_1-a_3)}\right\}$$

$$= a_1\,{}_3F_2\left[\begin{matrix}1+a_1-b_1, 1+a_1-b_2, -1\\ 1+a_1-a_2, 1+a_1-a_3\end{matrix};\right]$$

We consider therefore

$$F_n = \frac{(a_1)_n}{n!}\,{}_3F_2\left[\begin{matrix}1+a_1-b_1, 1+a_1-b_2, -n\\ 1+a_1-a_2, 1+a_1-a_3\end{matrix};\right]$$

which is annihilated by the operator

$$[\delta(\delta+a_1-a_2)(\delta+a_1-a_3)-(\delta+1+a_1-b_1)(\delta+1+a_1-b_2)(\delta-n)].$$

Writing this operator in the form

$$[P+(\delta-n)Q+(\delta-n)(\delta-n+1)R]$$

and evaluating P, Q and R we may show that F_n satisfies the difference relation (3) and therefore that

$$(4)\qquad (1-x)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n}{n!}\,{}_3F_2\left[\begin{matrix}1+a_1-b_1, 1+a_1-b_2, -n\\ 1+a_1-a_2, 1+a_1-a_3\end{matrix};\right]\frac{1}{(1-x)^n}$$

is a series solution of the original equation of the ${}_3F_2$ in descending powers of $(1-x)$. Other solutions are obviously obtained by permuting a_1, a_2, a_3. Also since (4) is a solution valid at the singularity $x=\infty$ it must be a constant multiple of the solution 5.1. (3), and comparing coefficients of x^{-a_1} we find

$$(5)\qquad (1-x)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n}{n!}\,{}_3F_2\left[\begin{matrix}1+a_1-b_1, 1+a_1-b_2, -n\\ 1+a_1-a_2, 1+a_1-a_3\end{matrix};\right]\frac{1}{(1-x)^n}$$

$$= (-)^{a_1}x^{-a_1}\,{}_3F_2\left[\begin{matrix}a_1, 1+a_1-b_1, 1+a_1-b_2\\ 1+a_1-a_2, 1+a_1-a_3\end{matrix};\frac{1}{x}\right].^{(1)}$$

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(1) c.f. CHAUNDY. 2. p. 62.(iii).

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